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Almost Sharp Fronts : Limit Equations for a Two-Dimensional Model with Fractional Derivatives

by

Zoe Atkins

Thesis

Submitted to the University of Warwick

for the degree of

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Contents

List of Figures

List of Tables

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Declarations

I declare that the work contained in this thesis is my own, except where indicated or cited. This thesis has not been submitted previously for a degree at this or another university.

Abstract

We consider the evolution of sharp fronts and almost-sharp fronts for the α-equation, where for an active scalar q the corresponding velocity is defined by $u = \nabla^{\perp}(-\Delta)^{-(2-\alpha)/2}q$ for $0 < \alpha < 1$. This system is introduced as a model interpolating between the two-dimensional Euler equation ($\alpha = 0$) and the surface quasi-geostrophic (SQG) equation ($\alpha = 1$).

The study of such fronts for the SQG equation was introduced as a natural extension when searching for potential singularities for the three-dimensional Euler equation due to similarities between these two systems, with sharp-fronts corresponding to vortex-lines in the Euler case (Constantin et al., 1994b).

Almost-sharp fronts were introduced in Córdoba et al. (2004) as a regularisation of a sharp front with thickness δ, with interest in the study of such solutions as $\delta \to 0$, in particular those that maintain their structure up to a time independent of δ. The construction of almost-sharp front solutions to the SQG equation is the subject of current work (Fefferman and Rodrigo, 2012). The existence of exact solutions remains an open problem.

For the α-equation we prove analogues of several known theorems for the SQG equations and extend these to investigate the construction of almost-sharp front solutions. Using a version of the Abstract Cauchy Kovalevskaya theorem (Safonov, 1995) we show for fixed $0 < \alpha < 1$, under analytic assumptions, the existence and uniqueness of approximate solutions and exact solutions for short-time independent of δ; such solutions take a form asymptotic to almost-sharp fronts. Finally, we obtain the existence and uniqueness of analytic almost-sharp front solutions.

Chapter 1

Introduction

The Navier-Stokes equation for incompressible fluid flows in $\mathbb{R}^n$ ($n = 2$ or 3) is given by:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u = -\nabla p + f, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

where, for position $x \in \mathbb{R}^n$ and time $t \geq 0$, the vector valued function $u(x,t) \in \mathbb{R}^n$ represents the advective fluid velocity, $\nu \geq 0$ the kinematic viscosity, and the scalar function $p(x,t)$ the pressure[1]. In the most general form of these equations as given in (1.1), f represents an external forcing term; in this thesis we will study a model that contains no such term. Equation (1.2) is the incompressibility condition. For a derivation of this equation see, for example, Chorin and Marsden (1979).

In this work we will study a system that interpolates between the two-dimensional Euler equation and the surface quasi-geostrophic equation. The Euler equations for incompressible flow (with no external forcing) are obtained by setting $\nu = 0$ in (1.1), that is:

[1]The standard gradient operator ∇ in spatial coordinates gives the following:

$$(u \cdot \nabla u)_i = \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j},$$

$$\nabla \cdot u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i},$$

and the Laplace operator in the spatial variables is defined by $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p, \tag{1.3}$$

$$\nabla \cdot u = 0. \tag{1.4}$$

The surface quasi-geostrophic (SQG) equation - a two-dimensional evolution equation for an active scalar q - is given by:

$$\frac{Dq}{Dt} = \partial_t q + u \cdot \nabla q = 0, \tag{1.5}$$

where the velocity u is defined in terms of a stream function ψ:

$$u = (u_1, u_2) = \nabla^{\perp}\psi \equiv \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right), \tag{1.6}$$

$$(-\Delta)^{1/2}\psi = q, \tag{1.7}$$

so that:

$$u = \nabla^{\perp}(-\Delta)^{-1/2} q. \tag{1.8}$$

For $x \in \mathbb{R}^2$ and $t \geq 0$, the scalar function $q(x, y, t)$ represents the potential temperature and u, as above, is the fluid velocity. For more information on stream functions see Acheson (1990). In particular, the incompressibility condition is automatically satisfied.

The SQG system is derived from more general equations modelling nonhomogeneous fluid flow in a rapidly rotating two-dimensional boundary of the three-dimensional half-space, with small Rossby and Ekman numbers (accounting for the rotation and dissipation), and with constant potential vorticity. For a detailed derivation of this system see Pedlosky (1987), and for an overview of the main steps required see Majda and Tabak (1996).

1.1 An Interpolation Model : The α-equation

The focus of this thesis is the study of sharp fronts and almost-sharp fronts for the α-equation. This system, in two spatial dimensions, is defined as follows:

$$\frac{Dq}{Dt} = \partial_t q + u \cdot \nabla q = 0, \tag{1.9}$$

where $q(x, y, t)$ is a scalar function and the associated velocity $u(x, y, t)$ is defined in terms of a stream function $\psi(x, y, t)$:

$$u = (u_1, u_2) = \nabla^{\perp}\psi \equiv \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right), \tag{1.10}$$

where:

$$(-\Delta)^{(2\,-\,\alpha)/2}\psi = q, \tag{1.11}$$

so that we recover:

$$u = \nabla^{\perp}(-\Delta)^{-(2\,-\,\alpha)/2}q. \tag{1.12}$$

We use $\nabla^{\perp}$ to denote the perpendicular gradient operator, $\nabla^{\perp} = (-\partial_y, \partial_x)$, and by the definition of the velocity given in (1.10), u automatically satisfies the incompressibility condition $\nabla \cdot u = 0$.

We consider this system posed on the two-dimensional cylinder, that is $(x, y) \in \mathbb{R}/\pi\mathbb{Z} \times \mathbb{R}$, for time $t \in [0, T]$. We focus only on the cases where $0 < \alpha < 1$; when $\alpha = 1$ we recover the two-dimensional surface quasi-geostrophic (SQG) equation, and for $\alpha = 0$ the two-dimensional Euler equation. The α-equation (1.9)-(1.12) represents an interpolation between these two extremes (Córdoba, Fonelos, Mancho and Rodrigo, 2005).

On the cylindrical domain, the stream function ψ is given by, on inversion of (1.11), the convolution of q with a kernel of the operator $(-\Delta)^{-(2\,-\,\alpha)/2}$. We introduce two forms of this kernel, alongside Riesz operators, in the next chapter; for more details see Córdoba, Fefferman and Rodrigo (2004).

1.2 Motivation

An important question for both the Navier-Stokes equation and the Euler equation (in two and three dimensions) is whether the solutions to these develop singularities in finite time. One candidate in the search for the formation of singularities has been the study of vortex patches - a simply connected and bounded region evolving in time, with constant vorticity in this region. It has already been shown that singularities do not occur in this situation (Chemin, 1991). A natural extension to this has been to study the vortex patch problem for the SQG equation (1.5)-(1.8)

(Córdoba et al., 2005). At present the existence of finite time singularities in this case also remains an open question.

The interest in studying potential singularity formation for the SQG equation are its connections to three-dimensional Euler. The SQG equation is presented as a two-dimensional system that is much simpler to study, yet retains many of the features of Euler. The main similarity between the two systems is the structure of the evolution for $\nabla^{\perp}\theta$ and the vorticity stream formulation for three-dimensional Euler, where the vorticity $\omega = \nabla \times u$:

$$\frac{D\nabla^{\perp}\theta}{Dt} = (\nabla u)\nabla^{\perp}\theta \quad \text{and} \quad \frac{D\omega}{Dt} = (\nabla u)\omega$$

respectively. This leads to several analogies between the two systems relating to the characterization of potential singular solutions, construction of the velocity and conserved quantities. Such comparisons are well documented in the literature, see for example Constantin, Majda and Tabak (1994a), Constantin, Majda and Tabak (1994b), Majda and Bertozzi (2002) and Rodrigo Diez (2004).

For the SQG equations, the search for singular solutions has been focused on sharp fronts - weak solutions that attain two constant values in two regions separated by a smooth curve φ (Rodrigo, 2004). The physical motivation for study of these solutions is the formation and evolution of weather fronts, discontinuities between masses of hot and cold air (Córdoba et al., 2004). Numerical evidence for the development of sharp fronts which become singular in finite time has been given in both Constantin et al. (1994a) and Constantin et al. (1994b). The interest in these particular solutions of the SQG equations is that they are analogous to the study of vortex lines for the three-dimensional Euler equation. A derivation of the sharp-front equation is contained in Rodrigo Diez (2004). This is a contour dynamics equation (CDE) describing the evolution of the smooth curve φ, and it was shown that smooth solutions to such a CDE exist in short time using a Nash-Moser type argument.

The techniques employed in the derivation of such a CDE are not available in the three-dimensional Euler case due to the singularity of the velocity in that case (this singularity is proportional to the distance from the curve φ, whereas the behaviour of the singularity in the SQG case is logarithmic), see Córdoba et al. (2004). One such method introduced in this paper for studying vortex lines is to consider solutions supported on a neighbourhood of this line (a vortex-tube) and obtain an evolution equation in the limit as the tube thickness aproaches 0. It is with this intention that the authors introduced the concept of almost-sharp fronts for the SQG equation; that is weak solutions of the SQG equation that are a regularisation of the sharp front, with a δ-neighbourhood around φ in which the solution changes from one constant to another. In Córdoba et al. (2004) it is shown that the evolution of the almost-sharp front behaves as the sharp-front equation (CDE) up to some error $O(\delta \log \delta)$, and in Fefferman, Luli and Rodrigo (2012) a curve, the spine, is introduced that satisfies the CDE up to an error $O(\delta \log \delta)$. These results rely on the existence of almost-sharp fronts.

Current study is now focused on the construction of these almost-sharp fronts for the SQG equation and their behaviour in the limit as the thickness of the front δ approaches 0. Of most interest are almost-sharp fronts that maintain their structure up to a time independent of δ (Fefferman et al., 2012). Here, the authors construct a family of almost-sharp fronts indexed by the thickness of the front δ, and derive an equation for the evolution of such a solution in the limit. It remains an open question as to whether smooth solutions to this equation exist due to the appearance of a Prandtl-like term; see for example Sammartino and Caflisch (1998a) and Sammartino and Caflisch (1998b).

A natural system to study in search of singularities is the α-equation as described in (1.9)-(1.12) and introduced in Córdoba et al. (2005). In this paper numerical evidence is outlined showing that sharp fronts for this equation develop singularities when $0 < \alpha \leq 1$, and a local existence result is given for the corresponding CDE. When $\alpha < 1$ the equation becomes simpler to study as the velocity is less singular than in the SQG case; in particular there is no logarithmic behaviour.

In this thesis we construct solutions to the α-equation; these will either be of a form asymptotic to almost-sharp fronts, and in the final case will be almost-sharp fronts. We prove a series of existence and uniqueness results for such solutions under analytic assumptions, using a version of the Abstract Cauchy Kovalevskaya (ACK) theorem (Safonov, 1995), for $0 < \alpha < 1$; these results are all new and remain open for the SQG case. Of particular interest is whether we can show existence of almost-sharp front solutions to the α-equation and recover almost-sharp front solutions to the SQG equations in the limit as $\alpha \to 1$.

We derive limit equations for almost sharp front solutions in the smooth case as seen for the SQG equations in Fefferman and Rodrigo (2012). For this case we discuss the existence of approximate solutions to the α-equation; although we obtain a simpler form of the limit equation when $0 < \alpha < 1$, at present the existence of solutions remains an open problem due to the presence of a Prandtl-like term.

An ideal result would be to prove existence of almost-sharp front solutions for some time independent of the thickness of the front. For the analytic case, we derive the limit equations and show that - in appropriately constructed function spaces - we are able to prove existence, and the uniqueness of approximate solutions (which are aymptotic to almost-sharp fronts) to the α-equation. We extend this result to show that we actually have exact solutions of the same form for some time independent of δ. The methods for proving existence in these cases have been introduced with the hope to extend these results to the $\alpha = 1$ case; we outline several problems with this following the presentation of these results.

The final result obtained in the thesis expands on this idea by the introduction of a second method for studying solutions to the α-equation. We show that we do indeed have existence of almost-sharp front solutions to the α-equation that maintain their structure in short-time independent of δ. This remains an open question for the SQG case.

1.3 Structure of the Thesis

The thesis is effectively broken into two parts; the first half of the thesis (Chapters 2, 3 and the first part of Chapter 4) contain analogues of results that have been shown for the SQG equations, presented in Rodrigo (2004), Córdoba et al. (2004), Fefferman et al. (2012), Fefferman and Rodrigo (2012) and Fefferman and Rodrigo (2011a), for the case when $0 < \alpha < 1$, and a discussion of the differences between the two systems. The second part of the thesis contains a series of existence results, to be outlined below, for analytic solutions to the α-equation, which remain open problems for the SQG case.

The main mathematical tools utilised throughout the thesis are properties of the convolution of two functions and the ACK theorem. An overview of these techniques, including a comparison of the standard Cauchy-Kovalevskaya theorem to the ACK theorem, are contained in Appendix A.

In the next chapter we introduce the α-equation in more detail and give the definition of the kernel that appears in the velocity term. Sharp fronts are introduced in §2.2, that is q which takes two constant values in two regions which change sharp over a boundary given by a smooth curve φ. Almost-sharp fronts, which are a regularisation of sharp-fronts across a δ-neighbourhood of the curve φ, are described in §2.3. Assuming the existence of such weak solutions, we derive a CDE for φ in the sharp-front case, the "Sharp-front Equation" (SFE), which will form a necessary requirement for all of the existence results presented in the analytic case. For almost sharp-front solutions we derive a corresponding evolution equation for φ, and for the 'spine' outlined in §2.4, which satisfy the SFE up to an error dependent on δ. These are analogues of the results in Rodrigo (2004), Córdoba et al. (2004), and Fefferman et al. (2012).

In Chapter 3, we construct a family of smooth almost-sharp fronts Ω indexed by the width of the front δ. For this specific family we find the associated limit equation; under the assumptions that φ satisfies the SFE and that Ω has Sobolev bounds independent of δ, we are able to take the formal limit as $\delta \to 0$. A discussion on the validity of these assumptions is contained in §3.1. Existence of solutions to the limit equation in the smooth case, which would be approximate solutions to the α-equation, is not yet known. This result is a analogue of that for the SQG equations as presented in Fefferman and Rodrigo (2012). We are however able to obtain such an existence results under analytic assumptions.

Chapter 4 contains the analogous result under the assumptions of analyticity. We first construct a family of solutions that are asymptotic to almost sharp fronts and, under a suitable change of coordinates derive the corresponding limiting equation obtained by taking the formal limit as $\delta \to 0$ (again assuming that the SFE is satisfied). The ACK theorem which is used in the existence results that follow is outlined in §4.3. Associating an IVP with the limit equation we are able to show, using the ACK theorem, that there exists a unique solution, in short time, to the limit equation which takes the form of an approximate almost-sharp front.

Of interest however are exact solutions to the α-equation whose time of existence is independent of the width of the front δ. This forms the remainder of the thesis.

Under the assumpitions of analyticity, in chapter 5 we prove the existence of a family of exact solutions to the α-equation; these will be of the same form introduced in Chapter 4. Applying the ACK theorem for function spaces defined within this chapter gives the existence and uniqueness result required. In addition we are also able to ensure that the existence holds for short-time independent of δ.

Chapter 6 contains a new method for constructing analytic almost-sharp front solutions to the α equation. Rewriting the α-equation under a new change of variables, a final application of the ACK theorem ensures the existence of such solutions in short-time, again independent of δ.

A discussion of the results and remaining open questions forms the basis for the conclusion in Chapter 7.

Chapter 2

Preliminary Results

In this chapter we present several results for particular solutions of the α-equation, namely those of sharp fronts and almost-sharp fronts, which will be introduced formally in §2.2 and §2.3 respectively. The theorems contained here are all analogues of existing results for the SQG equations (that is when $\alpha = 1$) as introduced in Rodrigo (2004), Rodrigo Diez (2004), Córdoba et al. (2004) and Fefferman et al. (2012).

Recall that we are considering the following system:

$$\frac{Dq}{Dt} = \partial_t q + u \cdot \nabla q = 0, \tag{2.1}$$

where u is given by:

$$u = \nabla^{\perp}(-\Delta)^{-(2-\alpha)/2} q, \tag{2.2}$$

posed on the two-dimensional cylinder $(x, y) \in \mathbb{R}/\pi\mathbb{Z} \times \mathbb{R}$, for time $t \in [0, T]$. We have periodic behaviour in the x-variable, and so any functions on this domain will be defined for $(x, y) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}$ and extended periodically to the whole plane. We restrict our attention to the case when $0 < \alpha < 1$, as detailed previously.

The velocity u can be written as the convolution with some kernel K such that $u = K * q$, where K is to be defined. The main differences that occur in the proofs that follow - when compared to those for the SQG equations - are due to differences in the structure of the kernel of the fractional Laplacian for different exponents, specifically the change in the singular behaviour of these at the origin as $\alpha \to 1$. A summary of the form of the kernels that we will employ throughout this thesis, and their corresponding behaviour, is discussed in the next section.

In §2.2 we define sharp fronts for the α-equation, described by a periodic curve φ, and derive a contour dynamics equation for the evolution of this curve - the "sharp front equation" - important for studying the limit equations in Chapters 3 and 4. In §2.3 we introduce a family of almost-sharp fronts, which are a regularisation of the sharp fronts, and show that the evolution equation for φ is the same as that of the sharp front up to some error term, with size to be determined based on the thickness of the front. An improvement on this result is given in §2.4. Following on from Fefferman et al. (2012), we define a special curve, the "spine", and show that this also evolves as the sharp front equation up to some error term, which is much smaller than in the more general case.

2.1 Definition of the Kernel

We first review the inverse fractional Laplacian in the two-dimensional plane indexed by β: $(-\Delta)^{-\beta}$ for $0 < \beta < 1$. When posed on the whole plane, for a given function f that is sufficiently smooth, the Reisz potentials as defined in Stein (1970) are as follows:

$$(I_\beta f)(x) = (-\Delta)^{-\beta} f(x) = \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^2} |x-y|^{2\beta-2} f(y) \mathrm{d}y, \tag{2.3}$$

where the constant is given by:

$$\gamma(\beta) = \frac{\pi 2^{2\beta} \Gamma(\beta)}{\Gamma(1-\beta)}, \tag{2.4}$$

and Γ is the Gamma function[1]. Note that we will make a particular choice of β below as required for the α-equation, that is $\beta = \frac{2-\alpha}{2}$.

The form of the fractional Laplacian on the two-dimensional cylindrical domain can be derived using standard reflection methods, see for example Evans (1998). The kernel for this operator, extending the case from $\alpha = 1$ in Rodrigo (2004), is given by:

$$\frac{\chi(u,v)}{(u^2+v^2)^{(2\,-\,2\beta)/2}} + \eta(u,v), \tag{2.5}$$

where $\chi(u,v) \in C_0^\infty$, $\chi(u,v) = 1$ for $|u-v| \leq r$ and $\operatorname{supp}\chi \subset \{|u-v| \leq R\}$ with $0 < r < R < \frac{1}{2}$, and $\eta(u,v) \in C_0^\infty$ with $\eta(0,0) = 0$. In addition, χ is periodic in the first argument with period π.

[1] For $\beta > 0$, $\Gamma(\beta) = \int_0^\infty e^{-t} t^{\beta-1} \mathrm{d}t$.

By commutativity of the differential operators that appear, we rewrite the velocity u (2.2) in the form $u = (-\Delta)^{-(2-\alpha)/2}\nabla^{\perp} q$. Posed on the two-dimensional cylinder, u can be written as a convolution with a kernel of the form (2.5). For $\alpha \in (0, 1)$ we have $u = K_\alpha * \nabla^{\perp} q$ where:

$$K_\alpha(u, v) = \frac{\chi(u, v)}{(u^2 + v^2)^{\alpha/2}} + \eta(u, v), \tag{2.6}$$

that is:

$$u(x, y) = \int_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} K_\alpha(x - \bar{x}, y - \bar{y})\nabla^{\perp}_{\bar{x},\bar{y}} q(\bar{x}, \bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}.$$

Remark 2.1. *By the symmetry methods used to derive the form of this kernel K_α on the cylindrical domain, the smooth function η is not uniquely defined; to simplify some of the calculations, without loss of generality, we set $\eta \equiv 0$.*

The above form for the velocity will be used for deriving the limit equation in the smooth case. The study of the limit equations in the analytic case and the subsequent existence results require an equivalent kernel that is analytic in both variables. We will use $u = \tilde{K}_\alpha * \nabla^{\perp} q$ where:

$$\tilde{K}_\alpha(u, v) = \frac{1}{(\cosh(v) - \cos(u))^{\alpha/2}}. \tag{2.7}$$

This kernel is automatically periodic in the first variable with period 2π, and has the same singularity type as K_α at the origin. To see this recall the Taylor expansions of the cosh and cos functions about the origin, which give:

$$\cosh(v) - \cosh(u) \approx \frac{1}{2}(u^2 + v^2) + h.o.t.$$

Remark 2.2. *Notice that the kernel K_α (2.6) corresponds to periodising the kernel obtained by inverting the α-Laplacian in $\mathbb{R}^2$ (which is analytic except for a singularity at the origin). The kernel $\tilde{K}_\alpha$ defined in (2.7) is also analytic except for a singularity at the origin (of the same order).*

The difference between these two kernels is analytic with fast decay at ∞. It is clear that adding any analytic function to (2.7) would not change any of the results that follow, as it is only the singularity that plays a role. In fact we could have stated the results contained within the thesis for any analytic kernel K with the same singular behaviour at the origin, as we never need the specific structure of the cosh *and* cos *functions. It has become customary for many authors to use the expression in (2.7) to fix ideas.*

When $\alpha < 1$ these kernels are integrable, in particular $K_\alpha \in L^1(\mathbb{R}/\pi\mathbb{Z} \times \mathbb{R})$ (and $\tilde{K}_\alpha \in L^1(\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R})$). For consistency, by applying a scaling to the latter that only alters any constants we are able to define a version of $\tilde{K}_\alpha$ of period π; in doing so we are then able to present all results on the domain $\mathbb{R}/\pi\mathbb{Z} \times \mathbb{R}$.

The theorems presented in the remainder of this chapter will be stated and proved only for K_α. The results for the second kernel are analogous.

2.2 The Sharp Front Equation

We now focus on sharp fronts for the α-equation. For the SQG equations the study of sharp fronts is presented as an analogue of the vortex patch problem for the two-dimensional Euler equations (Rodrigo, 2004). For the interpolation model we continue to study the evolution of a particular solution to the system (2.1)-(2.2) that takes constant values in two regions separated by a smooth curve (see Figure 2.1). We derive an equation for the evolution of such a curve.

In Rodrigo Diez (2004), in the study of the SQG equations, two different derivations for the corresponding "sharp front equation" are presented - the study of the equations in the limit approaching the curve and using weak solutions. Here, for (2.1)-(2.2) we derive the analogous equation using the latter; we assume existence of a weak solution to the α-equation for short time taking the form of a sharp front, and derive an evolution equation for the given boundary curve. The standard definition of a weak solution to (2.1) is as follows:

Definition 2.3. *A bounded function q is a weak solution for the α-equation (2.1) if for any $\phi \in C_0^\infty(\mathbb{R}/\pi\mathbb{Z} \times \mathbb{R} \times [0,T])$*

$$\iiint\limits_{[0,T]\times\mathbb{R}\times\mathbb{R}/\pi\mathbb{Z}} q(x,y,t)\partial_t\phi(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \iiint\limits_{[0,T]\times\mathbb{R}\times\mathbb{R}/\pi\mathbb{Z}} q(x,y,t)u(x,y,t)\cdot\nabla\phi(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0, \tag{2.8}$$

where u is defined in (2.2).

A sharp front is defined to be a solution that satifies two constant values in two regions, which change sharply over a boundary given by a smooth curve $y = \varphi(x,t)$. The α-equations are posed in a cylindrical domain, and so we define the curve φ to be periodic (of period π) in the first argument. We therefore consider solutions of the form:

$$q(x,y,t) = \begin{cases} \frac{1}{2} & \text{if} \quad y \geq \varphi(x,t) \\ -\frac{1}{2} & \quad \text{otherwise} \end{cases} \tag{2.9}$$

as illustrated.

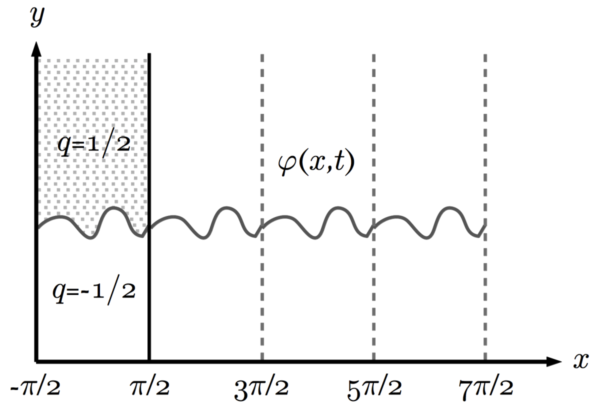

Figure 2.1: Sharp Front

Remark 2.4. *Note that in both Rodrigo Diez (2004) and Rodrigo (2004) the constants used for the different regions are instead 0 and 1. Without loss of generality we choose the values $-\frac{1}{2}$ and $\frac{1}{2}$; for later proofs within the paper this choice provides some cancellation of terms and enables us to simplify the calculations. For consistency we use this definition throughout the thesis. For the proofs that follow in this section, the change of constants only require elementary changes from the SQG case.*

We now study the evolution of $\varphi(x,t)$, and derive the following:

Theorem 2.5 (Sharp Front Equation 1)**.** *Let q be a weak solution of the α-equation as defined in (2.1), and let q be of the form (2.9). Then the function φ satisfies the equation:*

$$\frac{\partial \varphi}{\partial t}(x,t) = \int_{\mathbb{R}/\pi\mathbb{Z}} \frac{\frac{\partial \varphi}{\partial x}(x,t) - \frac{\partial \varphi}{\partial \bar{x}}(\bar{x},t)}{[(x-\bar{x})^2 + (\varphi(x,t)-\varphi(\bar{x},t))^2]^{\alpha/2}} \chi(x-\bar{x}, \varphi(x,t)-\varphi(\bar{x},t)) \mathrm{d}\bar{x}$$
$$+ \int_{\mathbb{R}/\pi\mathbb{Z}} \left(\frac{\partial \varphi}{\partial x}(x,t) - \frac{\partial \varphi}{\partial \bar{x}}(\bar{x},t) \right) \eta(x-\bar{x}, \varphi(x,t)-\varphi(\bar{x},t)) \mathrm{d}\bar{x}. \tag{2.10}$$

where, for consistency with existing work, the notation $\frac{\partial f}{\partial \bar{x}}(\bar{x},t) = \frac{\partial f}{\partial x}(\bar{x},t)$ is used to denote differentiation with respect to the spatial variable.

When considering the analytic case, as presented in Remark 2.2, we use an equivalent kernel. The same set of calculations gives a similar statement when we consider the kernel $\tilde{K}_\alpha$ (2.7):

Theorem 2.6 (Sharp Front Equation 2)**.** *Let q be a weak solution of the α-equation as defined in (2.1), and let q be of the form (2.9). Then the function φ satisfies the equation:*

$$\frac{\partial \varphi}{\partial t}(x,t) = \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{\frac{\partial \varphi}{\partial x}(x,t) - \frac{\partial \varphi}{\partial \bar{x}}(\bar{x},t)}{(\cosh(\varphi(x,t) - \varphi(\bar{x},t)) - \cos(x - \bar{x}))^{\alpha/2}} \mathrm{d}\bar{x}. \tag{2.11}$$

Both forms of the sharp front equation will be utilised when deriving the limit equations in Chapters 3 and 4.

The proof of Theorem 2.5, up to the choice of constants, remains the same as that presented in Rodrigo (2004). For completeness and in order to introduce some of the notation, we give an overview of the details here. The methods used in Rodrigo (2004) for the SQG equations in effect show that if we have a system as in (2.1) with the function u of the form $u = \nabla^\perp K * q$ for some kernel K, then assuming that K is regular enough that all integrals make sense, the proof can be generalised further.

Proof of Theorem 2.5. Note that for all $\alpha < 1$ it follows from the definition that $\nabla^\perp K_\alpha \in L^1$. Given $u = \nabla^\perp K_\alpha * q$, since $q \in L^\infty$ an application of Young's inequality for convolutions (Theorem A.1) gives $u \in L^1$ and so all of the integrals below are well defined. We first set:

$$I = \{(x,y,t) : y \geq \varphi(x,t)\}, \quad II = \{(x,y,t) : y < \varphi(x,t)\},$$

where $q \equiv \frac{1}{2}$ on I and $q \equiv -\frac{1}{2}$ on II. The outward unit normals for each region, as required for integration by parts, are respectively:

$$\nu^I(x,y,t) = (\nu^I_x, \nu^I_y, \nu^I_t) = \frac{(\partial_x\varphi, -1, \partial_t\varphi)}{(1 + (\partial_x\varphi)^2 + (\partial_t\varphi)^2)^{1/2}},$$
$$\nu^{II}(x,y,t) = (\nu^{II}_x, \nu^{II}_y, \nu^{II}_t) = \frac{(-\partial_x\varphi, 1, -\partial_t\varphi)}{(1 + (\partial_x\varphi)^2 + (\partial_t\varphi)^2)^{1/2}}.$$

For the form of q as given in (2.9), we substitute q into the definition of the weak solution (2.8) and derive an equation for the evolution of the curve φ. Evaluating the first term in this definition, an integration by parts gives:

$$\iiint\limits_{[0,T]\times\mathbb{R}\times\mathbb{R}/\pi\mathbb{Z}} q\partial_t\phi \mathrm{d}x\mathrm{d}y\mathrm{d}t = \iiint\limits_{y\geq\varphi(x,t)} \frac{1}{2}\partial_t\phi \mathrm{d}x\mathrm{d}y\mathrm{d}t - \iiint\limits_{y<\varphi(x,t)} \frac{1}{2}\partial_t\phi \mathrm{d}x\mathrm{d}y\mathrm{d}t$$

$$= \frac{1}{2}\iint\limits_{y=\varphi(x,t)} \phi\nu_t^I(1+(\partial_x\varphi)^2+(\partial_t\varphi)^2)\mathrm{d}x\mathrm{d}t - \frac{1}{2}\iint\limits_{y=\varphi(x,t)} \phi\nu_t^{II}(1+(\partial_x\varphi)^2+(\partial_t\varphi)^2)\mathrm{d}x\mathrm{d}t$$

$$= \frac{1}{2}\iint\limits_{y=\varphi(x,t)} \phi\partial_t\varphi \mathrm{d}x\mathrm{d}t - \frac{1}{2}\iint\limits_{y=\varphi(x,t)} \phi(-\partial_t\varphi)\mathrm{d}x\mathrm{d}t$$

$$= \iint\limits_{y=\varphi(x,t)} \phi\partial_t\varphi \mathrm{d}x\mathrm{d}t. \tag{2.12}$$

Considering only the spatial integration to begin with, we study the second term in (2.8). Introducing limits and on integrating by parts we obtain:

$$\iint\limits_{\mathbb{R}\times\mathbb{R}/\pi\mathbb{Z}} qu\cdot\nabla\phi \mathrm{d}x\mathrm{d}y = \frac{1}{2}\lim_{\delta\to 0}\int\limits_{y=\varphi(x,t)+\delta} u\phi\cdot\begin{pmatrix}\partial_x\varphi\\ -1\end{pmatrix}\mathrm{d}x$$

$$- \frac{1}{2}\lim_{\delta\to 0}\int\limits_{y=\varphi(x,t)-\delta} u\phi\cdot\begin{pmatrix}-\partial_x\varphi\\ 1\end{pmatrix}\mathrm{d}x, \tag{2.13}$$

where:

$$u\phi\cdot\begin{pmatrix}\partial_x\varphi\\ -1\end{pmatrix} = \frac{1}{2}\phi(x,y,t)\int\limits_{\bar{y}\geq\varphi(\bar{x},t)} \nabla^{\perp}_{x,y}K_\alpha(x-\bar{x},y-\bar{y})\cdot\begin{pmatrix}\partial_x\varphi\\ -1\end{pmatrix}\mathrm{d}\bar{x}\mathrm{d}\bar{y}$$

$$- \frac{1}{2}\phi(x,y,t)\int\limits_{\bar{y}<\varphi(\bar{x},t)} \nabla^{\perp}_{x,y}K_\alpha(x-\bar{x},y-\bar{y})\cdot\begin{pmatrix}\partial_x\varphi\\ -1\end{pmatrix}\mathrm{d}\bar{x}\mathrm{d}\bar{y}$$

$$= A_1 + A_2.$$

Using the chain rule and an application of the divergence theorem, as in Rodrigo (2004), we have:

$$
\begin{aligned}
A_1 &= \frac{1}{2}\phi(x,y,t) \int\limits_{\bar{y}\geq\varphi(\bar{x},t)} -\nabla^{\perp}_{\bar{x},\bar{y}} K_\alpha(x-\bar{x},y-\bar{y}) \cdot \begin{pmatrix} \partial_x\varphi \\ -1 \end{pmatrix} \mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&= \frac{1}{2}\phi(x,y,t) \int\limits_{\bar{y}\geq\varphi(\bar{x},t)} \nabla_{\bar{x},\bar{y}} K_\alpha(x-\bar{x},y-\bar{y}) \cdot \begin{pmatrix} 1 \\ \partial_x\varphi \end{pmatrix} \mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&= \frac{1}{2}\phi(x,y,t) \int\limits_{\bar{y}\geq\varphi(\bar{x},t)} \nabla_{\bar{x},\bar{y}} \cdot \begin{pmatrix} K_\alpha(x-\bar{x},y-\bar{y}) \\ \partial_x\varphi K_\alpha(x-\bar{x},y-\bar{y}) \end{pmatrix} \mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&= \frac{1}{2}\phi(x,y,t) \int\limits_{\bar{y}=\varphi(\bar{x},t)} K_\alpha(x-\bar{x},y-\bar{y}) \begin{pmatrix} 1 \\ \partial_x\varphi \end{pmatrix} \cdot \begin{pmatrix} \partial_{\bar{x}}\varphi \\ -1 \end{pmatrix} \mathrm{d}\bar{x} \\
&= \frac{1}{2}\phi(x,y,t) \int\limits_{\bar{y}=\varphi(\bar{x},t)} K_\alpha(x-\bar{x},y-\bar{y}) \left(\partial_{\bar{x}}\varphi - \partial_x\varphi\right) \mathrm{d}\bar{x}
\end{aligned}
$$

and similarly for A_2. This gives:

$$
u\phi \cdot \begin{pmatrix} \partial_x\varphi \\ -1 \end{pmatrix} = \phi(x,y,t) \int\limits_{\bar{y}=\varphi(\bar{x},t)} \left(\frac{\partial\varphi}{\partial\bar{x}} - \frac{\partial\varphi}{\partial x}\right) K_\alpha(x-\bar{x},y-\bar{y})\mathrm{d}\bar{x}, \tag{2.14}
$$

and taking limits as $\delta \to 0$ in (2.13) we have:

$$
\iint\limits_{\mathbb{R}\times\mathbb{R}/\pi\mathbb{Z}} qu\cdot\nabla\phi\mathrm{d}x\mathrm{d}y = \int\limits_{y=\varphi(x,t)} \phi(x,y,t) \int\limits_{\bar{y}=\varphi(\bar{x},t)} \left(\frac{\partial\varphi}{\partial\bar{x}} - \frac{\partial\varphi}{\partial x}\right) K_\alpha(x-\bar{x},y-\bar{y})\mathrm{d}\bar{x}\mathrm{d}x.
$$

Combining this with (2.12) gives:

$$
\begin{gathered}
\iint\limits_{y=\varphi(x,t)} \phi\partial_t\varphi\mathrm{d}x\mathrm{d}t \\
+ \int\limits_{y=\varphi(x,t)} \phi(x,y,t) \int\limits_{\bar{y}=\varphi(\bar{x},t)} \left(\frac{\partial\varphi}{\partial\bar{x}} - \frac{\partial\varphi}{\partial x}\right) K_\alpha(x-\bar{x},y-\bar{y})\mathrm{d}\bar{x}\mathrm{d}x\mathrm{d}t = 0
\end{gathered}
$$

and we obtain:

$$\partial_t\varphi = \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \left(\frac{\partial\varphi}{\partial x} - \frac{\partial\varphi}{\partial\bar{x}}\right) K_\alpha(x-\bar{x}, y-\varphi(\bar{x},t))\mathrm{d}\bar{x},$$

which is precisely the sharp front equation. □

Given an initial condition $\varphi(x,0) = \varphi_0(x)$ we can consider an initial value problem for the equation (2.10). For the SQG equations ($\alpha = 1$), it has been shown that for smooth, periodic functions φ_0 the system had a unique smooth solution for a small time. This has been proved in Rodrigo Diez (2004) using a Nash-Moser argument. For other values of α, local existence and uniqueness of smooth solutions to the corresponding IVP has been outlined in Córdoba et al. (2005) using the same argument. In Fefferman and Rodrigo (2011b) the authors show that, on application of a version of the Abstract Cauchy-Kovalevskaya theorem that given an IVP for the sharp-front equation with analytic initial data, there exists a unique analytic solution in short-time.

For $0 < \alpha < 1$ Gancedo (2008) defines a more general almost sharp front, that is:

$$q(x_1,x_2,t) = \begin{cases} q_1 & \Omega(t) \\ q_2 & \mathbb{R}^2/\Omega(t) \end{cases}$$

where q_1 and q_2 are constant and the boundary is parametrised by $\partial\Omega(t) = \{x(\gamma,t) = (x_1(\gamma,t), x_2(\gamma,t)) : \gamma \in [-\pi,\pi]\}$. An equivalent CDE to (2.10) and (2.11) is derived and the author shows that, under additional assumptions on the boundary[2], for $x_0(\gamma) \in H^k(\mathbb{T})$ where $k \geq 3$, then there exists a time $T > 0$ such that there exists a unique solution to the associated IVP in $C^1([0,T]; H^k(\mathbb{T}))$.

2.3 Almost-Sharp Fronts

We now turn our attention to the evolution of almost-sharp fronts for the α-equation (2.1)-(2.2). These are a regularisation of sharp fronts as introduced previously, and are weak solutions of the α-equation (see Definition 2.3) that take two constant values which change in a transition strip of width 2δ; these solutions have large gradient of order $\frac{1}{\delta}$. The transition layer is defined as a δ-neighbourhood of a given

[2]For $F(x) =: \frac{|\eta|}{|x(\gamma,t)-x(\gamma-\eta,t)|}$ $\forall\gamma,\eta \in [-\pi,\pi]$ and $F(x)(\gamma,0,t) = \frac{1}{|\partial_\gamma x(\gamma,t)|}$ it is required that $F(x_0)(\gamma,\eta) < \infty$.

smooth curve $y = \varphi(x,t)$. We remain in the cylindrical case and define almost-sharp fronts to be weak solutions of the form:

$$q(x,y,t) = \begin{cases} \frac{1}{2} & \text{if} \quad y \geq \varphi(x,t) + \delta \\ \text{bounded} & \text{if} \quad |\varphi(x,t) - y| \leq \delta \\ -\frac{1}{2} & \text{if} \quad y \leq \varphi(x,t) - \delta \end{cases} \tag{2.15}$$

where φ is periodic in x; these are illustrated in Figure 2.2.

In this form we derive an evolution equation for the curve $\varphi(x,t)$; we show that this curve in fact satisfies the sharp front equation (Theorems 2.5 and 2.6) up to some error term of order δ (the same error for which the function φ is defined in (2.15)).

This problem was introduced in Córdoba et al. (2004) for the SQG equations, $\alpha = 1$, and the authors show that the curve φ satisfies the corresponding sharp-front equation up to some error of size $\delta \log \delta$. The following result shows that when $\alpha < 1$ we obtain a better estimate due to the singularity of the kernel in this case.

Theorem 2.7. *Let q be a weak solution of the α-equation (2.8), and let q be of the form (2.15), then the curve φ satisfies the following equation:*

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(x,t) = & \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{\frac{\partial \varphi}{\partial x}(x,t) - \frac{\partial \varphi}{\partial u}(u,t)}{[(x-u)^2 + (\varphi(x,t) - \varphi(u,t))^2]^{\alpha/2}} \chi(x-u, \varphi(x,t) - \varphi(u,t)) \mathrm{d}u \\ & + \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \left(\frac{\partial \varphi}{\partial x}(x,t) - \frac{\partial \varphi}{\partial u}(u,t) \right) \eta(x-u, \varphi(x,t) - \varphi(u,t)) \mathrm{d}u + Error, \end{aligned}$$

where $|Error| \leq C\delta$ with C depending only on $\|q\|_\infty$ and $\|\nabla\varphi\|_\infty$.

We introduce the notation $X = O(Y)$ when $|X| \leq C|Y|$, where in the following the constant C will depend only on $\|q\|_{L^\infty}$, $\|\nabla\varphi\|_{L^\infty}$ and $\|\phi\|_{C^1}$ which are independent of δ. Note that we use the standard norms in each of these function spaces, for details see Evans (1998).

Remark 2.8. *The proof of Theorem 2.7 below follows the techniques employed in Córdoba et al. (2004); the main differences that arise in the proof are due to the singularity of the kernel K_α. When $\alpha = 1$ the singularity that occurs requires the study of functions of class $L \log L$ (see for example Stein (1993)) in order to complete some of the estimates, leading to the logarithmic behaviour as previously noted. When $\alpha < 1$, $K_\alpha \in L^1$ and so many of the estimates that occur in the SQG case can be simplified.*

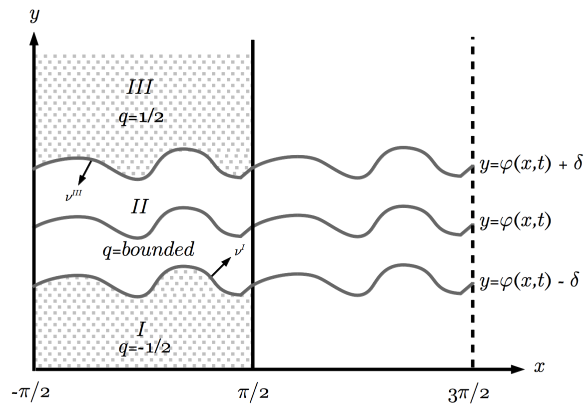

Figure 2.2: Almost - Sharp Front

Proof of Theorem 2.7. We first define the three regions as introduced in (2.15); these are illustrated in Figure 2.2. Let:

$$I = \{(x,y) : y \leq \varphi(x,t) - \delta\}, \quad II = \{(x,y) : |y - \varphi(x,t)| < \delta\},$$
$$III = \{(x,y) : y \geq \varphi(x,t) + \delta\},$$

with the corresponding outward unit normals for regions I and III as follows:

$$\nu^{I}(x,y,t) = (\nu^{I}_x, \nu^{I}_y, \nu^{I}_t) = \frac{(-\partial_x\varphi, 1, -\partial_t\varphi)}{[1 + (\partial_x\varphi)^2 + (\partial_t\varphi)^2]^{1/2}},$$
$$\nu^{III}(x,y,t) = (\nu^{III}_x, \nu^{III}_y, \nu^{III}_t) = \frac{(\partial_x\varphi, -1, \partial_t\varphi)}{[1 + (\partial_x\varphi)^2 + (\partial_t\varphi)^2]^{1/2}}.$$

In order to simplify the calculations in the arguments that follow, we prove this theorem for the almost-sharp front defined as in Córdoba et al. (2004):

$$q(x,y,t) = \begin{cases} 1 & \text{if} \quad y \geq \varphi(x,t) + \delta \\ \text{bounded} & \text{if} \quad |\varphi(x,t) - y| \leq \delta \\ 0 & \text{if} \quad y \leq \varphi(x,t) - \delta \end{cases} \tag{2.16}$$

We substitute this definition into that of the weak solutions (2.8). The same estimate, presented in Theorem 2.7, holds for the previous definition (2.15), which we employ throughout the thesis.

Remark 2.9. *For simplicity in the following proof we use the almost-sharp front* *defined in (2.16). To see that the proof also holds for the previous definition (2.15), let $\tilde{q}$ be an almost sharp front of the form in (2.15). Then for q as in (2.16) we have $\tilde{q} = q + \frac{1}{2}$. We note that the corresponding velocities are the same, that is:*

$$u = K * \nabla^{\perp} q = K * \nabla^{\perp}(q + \frac{1}{2}) = K * \nabla^{\perp} \tilde{q},$$

and so all integral estimates presented within the proof of this theorem hold for the almost-sharp front in (2.15).

Note that:

$$\iiint\limits_{I \times [0,T]} q(x,y,t)\partial_t \phi(x,y,t) \mathrm{d}x\mathrm{d}y\mathrm{d}t + \iiint\limits_{I \times [0,T]} q(x,y,t)u(x,y,t) \cdot \nabla \phi(x,y,t) \mathrm{d}x\mathrm{d}y\mathrm{d}t = 0$$

and so we study in detail the contributions from the other two regions.

We first show that the integrals over region II contribute to the error terms, that is they are $O(\delta)$. Note that the area of this region is $O(\delta)$ with dependence on $\|\nabla \varphi\|_{L^\infty}$. We have:

$$\iiint\limits_{II \times [0,T]} q(x,y,t)\partial_t \phi(x,y,t) \mathrm{d}x\mathrm{d}y\mathrm{d}t \leq \delta \|q\|_{L^\infty} \|\phi\|_{C^1}.$$

For $u = \nabla^{\perp} K_\alpha * q$ with $\nabla^{\perp} K_\alpha \in L^1$ and q bounded, it follows, by an application of Young's inequality for convolutions, that $\|u\|_\infty \leq C\|q\|_\infty$ giving:

$$\iiint\limits_{II \times [0,T]} q(x,y,t)u(x,y,t) \cdot \nabla \phi(x,y,t) \mathrm{d}x\mathrm{d}y\mathrm{d}t \leq \delta \|q\|^2_{L^\infty} \|\phi\|_{C^1}.$$

In order to calculate the integrals from region III, we introduce a decomposition for u; $u = u_{II} + u_{III} = \nabla^{\perp} K_\alpha * q \mathbb{1}_{II} + \nabla^{\perp} K_\alpha * \mathbb{1}_{III}$. Note that u_{II} and u_{III} are divergence free. Then:

$$\iiint\limits_{III \times [0,T]} q(x,y,t)u_{II}(x,y,t) \cdot \nabla \phi(x,y,t) \mathrm{d}x\mathrm{d}y\mathrm{d}t \leq \delta \|q\|^2_{L^\infty} \|\phi\|_{C^1}$$

as previously. It remains to determine:

$$\iiint\limits_{III\times[0,T]} q(x,y,t)\partial_t\phi(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t + \iiint\limits_{III\times[0,T]} q(x,y,t)u_{III}(x,y,t)\cdot\nabla\phi(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

The former can be calculated using integration by parts as in the proof of Theorem 2.5. That is:

$$\begin{aligned}\iiint\limits_{III\times[0,T]} q(x,y,t)\partial_t\phi(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t &= \iiint\limits_{y\geq\varphi(x,t)+\delta} \partial_t\phi(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= \iint\limits_{y=\varphi(x,t)+\delta} \phi(x,y,t)\partial_t\varphi\mathrm{d}x\mathrm{d}t,\end{aligned}$$

and for the latter, with u_{III} divergence free, considering spatial integration only to begin with:

$$\begin{aligned}\iint\limits_{III} u_{III}\cdot\nabla\phi\mathrm{d}x\mathrm{d}y &= \lim_{\epsilon\to0^+} \iint\limits_{y\geq\varphi(x,t)+\delta+\epsilon} u_{III}\cdot\nabla\phi\mathrm{d}x\mathrm{d}y \\ &= \lim_{\epsilon\to0^+} \int\limits_{y=\varphi(x,t)+\delta+\epsilon} u_{III}\phi(x,y,t)\cdot\begin{pmatrix}\partial_x\varphi\\-1\end{pmatrix}\mathrm{d}x,\end{aligned}$$

where as in §2.2:

$$u_{III}\phi(x,y,t)\cdot\begin{pmatrix}\partial_x\varphi\\-1\end{pmatrix} = \phi(x,y,t)\int\limits_{\bar{y}=\varphi(\bar{x},t)+\delta} K_\alpha(x-\bar{x},y-\bar{y})\left(\partial_{\bar{x}}\varphi-\partial_x\varphi\right)\mathrm{d}\bar{x}.$$

Taking limits as $\epsilon\to0$ and combining the results :

$$\begin{gathered}\iint\limits_{y=\varphi(x,t)+\delta} \phi(x,y,t)\partial_t\varphi\mathrm{d}x\mathrm{d}t \\ + \int\limits_{y=\varphi(x,t)+\delta} \phi(x,y,t)\int\limits_{\bar{y}=\varphi(\bar{x},t)+\delta} K_\alpha(x-\bar{x},y-\bar{y})\left(\partial_{\bar{x}}\varphi-\partial_x\varphi\right)\mathrm{d}\bar{x}\mathrm{d}x\mathrm{d}t \\ +O(\delta)=0\end{gathered}$$

and we obtain:

$$\partial_t\varphi(x,t) = \int\limits_{\mathbb{R}/\pi\mathbb{Z}} K_\alpha(x-\bar{x},\varphi(x,t)-\varphi(\bar{x},t))\,(\partial_{\bar{x}}\varphi - \partial_x\varphi)\,\mathrm{d}\bar{x}$$

as required. In order to adapt the proof for the almost-sharp front defined in (2.15) we use the same techniques alongside the continuity of the integrands and the corresponding decomposition $u = u_I + u_{II} + u_{III} = \frac{1}{2}\nabla^\perp K_\alpha * \mathbb{1}_I + \nabla^\perp K_\alpha * q\mathbb{1}_{II} + \frac{1}{2}\nabla^\perp K_\alpha * \mathbb{1}_{III}$. □

2.4 The Spine of an Almost-Sharp Front

In Fefferman et al. (2012), given a solution of the SQG equation ($\alpha = 1$) that is locally constant outside a δ-neighbourhood of a given curve φ that evolves with time, the authors define the concept of the 'spine'. They show that given an almost-sharp front weak solution of the SQG equation, there exists an associated curve (the 'spine') that can be explicitly defined and evolves as the sharp front equation up to some error of size $\delta^2|\log\delta|$. This improves the result of Córdoba et al. (2004) as the spine is shown to evolve up to an error smaller than δ as was previously given (see §2.3). The spine is defined in the transition layer $|\varphi(x,t) - y| < \delta$ and is constructed using an argument that estimates the difference between two measures; a delta function on the curve being constructed, μ, and $\nabla^\perp q \mathrm{d}x\mathrm{d}y$. The construction is independent of the form of u and so we may extend the definition to the case $0 < \alpha < 1$; we refer the reader to Fefferman et al. (2012) for the details.

In this section we give an analogous result for the α-equation for values of $0 < \alpha < 1$. We show that the associated spine for this equation evolves as the sharp front equation (2.10) up to an error of order δ^2, giving an improvement on the result presented in §2.3. We first give the definition of the spine and an extended definition of almost sharp fronts as required for the proof.

Definition 2.10. *For a function q of the form (2.15) we define the spine (in the region $|\varphi(x,t) - y| < \delta$), $y = \mu(x)$, by*

$$\int\limits_{\mathbb{R}} q_y(x,y)(y-\mu(x))\mathrm{d}y = 0 \quad \forall x. \tag{2.17}$$

Definition 2.11. *Assume that $q(x,y,t)$ is a weak solution (definition 2.3) of the α-equation defined for $(x,y,t) \in \mathbb{R}/\pi\mathbb{Z} \times \mathbb{R} \times [0,T]$. Let $\mu(x,t)$ be the curve defined at every time slice by the condition*

$$\int_{\mathbb{R}} q_y(x,y,t)(y-\mu(x,t))\mathrm{d}y = 0 \quad \forall x. \tag{2.18}$$

Assume that q is of the form

$$q(x,y) = \begin{cases} \frac{1}{2} & \textit{if} \quad y \geq \mu(x)+\delta \\ \textit{smooth} & \textit{if} \quad |\mu(x)-y| < \delta \\ -\frac{1}{2} & \textit{if} \quad y \leq \mu(x)-\delta \end{cases} \tag{2.19}$$

and that it satisfies the growth conditions

$$|\partial_x^\beta q| \leq c_\beta \delta^{-|\beta|} \quad \forall |\beta| \leq 2. \tag{2.20}$$

A function q with these properties above will be called an almost-sharp front.

Remark 2.12. *The spine condition (2.17) also gives the property:*

$$\int_{\varphi(x,t)-\delta}^{\varphi(x,t)+\delta} q(x,y,t)\mathrm{d}y = 0 \tag{2.21}$$

using integration by parts and shown in Fefferman et al. (2012).

This definition of an almost-sharp front requires more regularity plus additional growth conditions than as defined in §2.3. The growth conditions were introduced in Fefferman et al. (2012) for construction of the spine. Note that this is the only section in which we use this definition; for the remaining chapters an almost-sharp front will be as defined in §2.3. We prove the following:

Theorem 2.13. *Let q be an almost-sharp front for the α-equation as in definition 2.11 and let μ be its corresponding spine as introduced in definition 2.10. Then for every test function $\phi(x,t)$, the spine satisfies:*

$$\begin{aligned} \iint_{\mathbb{R}\times\mathbb{R}/\pi\mathbb{Z}} \phi(x,t)\Big[\frac{\partial\mu}{\partial t}(x,t) - \\ \int_{\mathbb{R}/\pi\mathbb{Z}} K_\alpha(x-u,\mu(x,t)-\mu(u,t))\Big(\frac{\partial\mu}{\partial x}(x,t) - \frac{\partial\mu}{\partial u}(u,t)\Big)\mathrm{d}u\Big]\mathrm{d}x\mathrm{d}t \\ = Error, \end{aligned} \tag{2.22}$$

where $|Error| \leq C\delta^2$ and C depends only on the constants c_β as in definition 2.11.

We will utilise the following result from Fefferman et al. (2012):

Corollary 2.14. *For $\mu(x,t)$ as defined in Definition 2.11, then for every $\phi(x,y)$ with $|\nabla^2\phi| \leq M$:*

$$\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} \phi(x,y,t)\nabla^{\perp}q(x,y,t)\mathrm{d}x\mathrm{d}y = -\int\limits_{y=\mu(x)} \phi(x,y)(1,\frac{\partial\mu}{\partial x}(x,t))\mathrm{d}x + O(M\delta^2).$$

Proof of Theorem 2.13 The sharp front equation (2.10) is given by:

$$\mu_t(x,t) = \int_{\mathbb{R}/\pi\mathbb{Z}} \left[\frac{\partial\mu}{\partial x}(x,t) - \frac{\partial\mu}{\partial\bar{x}}(u,t)\right] K_\alpha(x-u,\mu(x,t)-\mu(u,t))\mathrm{d}u \tag{2.23}$$

and given a weak solution q of the α-equation with a test function $\phi(x,y,t)$ we have:

$$\iiint\limits_{[0,T]\times\mathbb{R}\times\mathbb{R}/\pi\mathbb{Z}} q(x,y,t)\partial_t\phi(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{2.24}$$

$$+ \iiint\limits_{[0,T]\times\mathbb{R}\times\mathbb{R}/\pi\mathbb{Z}} q(x,y,t)u(x,y,t)\cdot\nabla\phi(x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t = 0. \tag{2.25}$$

We aim to show (2.22) with error $O(\delta^2)$. As remarked in Fefferman and Rodrigo (2011b), (2.22) contains only test functions that depend on x and t: we can assume that this is the case near to the spine. So while (2.24) and (2.25) are true for general test functions $\phi(x,y,t)$ we only need to consider functions that are constant in y near to the curve μ. This family of test functions will suffice to prove the result in (2.22). We sketch only an outline of the proof here and refer to the details in Fefferman et al. (2012).

Firstly the term (2.24), by the same method in that paper, is precisely:

$$\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} \phi(x,t)\mu_t(x,t)\mathrm{d}x\mathrm{d}t + O(\delta^2) \tag{2.26}$$

using (2.21). This is independent of the kernel chosen.

The differences in the proof when $\alpha < 1$, compared to that for the SQG case, occur when considering the estimates on the term (2.25). Here the singularity of the kernel K_α, defined in §2.1, ensures that we have no logarithmic behaviour. We show that (2.25) is equal to:

$$-\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} \phi(x,t) \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \left[\frac{\partial \mu}{\partial x}(x,t) - \frac{\partial \mu}{\partial \bar{x}}(\bar{x},t)\right] K_\alpha(x-\bar{x}, \mu(x,t)-\mu(\bar{x},t))\mathrm{d}\bar{x} \;+\; O(\delta^2). \tag{2.27}$$

Analogously to the proof of Theorem 2.3, we write (2.25) as a sum of integrals taken over the three domains I, II and III. Using the same notation as in Fefferman et al. (2012) we consider only the spatial integration of this term and study, using $u = \nabla^\perp K_\alpha * q$, the following function $B(t)$. Introducing the notation $\boldsymbol{x} = (x, y) \equiv (x_1, x_2)$ and $\boldsymbol{u} = (u_1, u_2)$ in order to simplify the following, we define:

$$\begin{aligned} B(t) = & \iint\limits_{\boldsymbol{x},\boldsymbol{u}\in I\cup III} \nabla^\perp_{\boldsymbol{x}} K_\alpha(\boldsymbol{x}-\boldsymbol{u})q(\boldsymbol{u})\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{u} \\ + & \iint\limits_{\boldsymbol{x}\in I\cup III,\boldsymbol{u}\in II \;\; or \;\; \boldsymbol{u}\in I\cup III,\boldsymbol{x}\in II} \nabla^\perp_{\boldsymbol{x}} K_\alpha(\boldsymbol{x}-\boldsymbol{u})q(\boldsymbol{u})\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{u} \\ + & \iint\limits_{\boldsymbol{x},\boldsymbol{u}\in II} \nabla^\perp_{\boldsymbol{x}} K_\alpha(\boldsymbol{x}-\boldsymbol{u})q(\boldsymbol{u})\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{u} \\ = & B_{outer} + B_{cross} + B_{inner}. \end{aligned} \tag{2.28}$$

The authors show, using integration by parts and symmetrizing some of the integrals, that this is equivalent to studying just three terms:

$$\iint\limits_{\boldsymbol{x}\in I\cup III,\boldsymbol{u}\in II} [\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) - \nabla_{\boldsymbol{u}}\phi(\boldsymbol{u})]\nabla^\perp_{\boldsymbol{x}} K_\alpha(\boldsymbol{x}-\boldsymbol{u})q(\boldsymbol{x})q(\boldsymbol{u})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{u}, \tag{B1}$$

$$\iint\limits_{\boldsymbol{x},\boldsymbol{u}\in II} K_\alpha(\boldsymbol{x}-\boldsymbol{u})[\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) - \nabla_{\boldsymbol{u}}\phi(\boldsymbol{u})]q(\boldsymbol{u})\nabla^\perp_{\boldsymbol{x}} q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{u}, \tag{B2}$$

$$\iint\limits_{\boldsymbol{x},\boldsymbol{u}\in I\cup III} \nabla^\perp_{\boldsymbol{x}} K_\alpha(\boldsymbol{x}-\boldsymbol{u})q(\boldsymbol{u})\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{u}, \tag{B3}$$

and showing that (B1) and (B2) are $O(\delta^2)$, and that (B3) is given by:

$$\int\limits_{x_2=\mu(x_1)} \int\limits_{u_2=\mu(u_1)} \phi(x_1,x_2)\left(\frac{\partial\mu}{\partial u_1}(u_1,t) - \frac{\partial\mu}{\partial x_1}(x_1,t)\right)K_\alpha(\boldsymbol{x}-\boldsymbol{u})\mathrm{d}x_1\mathrm{d}u_1 + O(\delta^2), \tag{2.29}$$

which are precisely the estimates needed to show (2.22).

On integrating by parts, using $\nabla^\perp q = 0$ in $I \cup III$, we have:

$$\begin{aligned}
(\mathrm{B1}) = \int\limits_{\boldsymbol{u}\in II} \left[\sum_{\sigma=\pm1} \int\limits_{x_2=\mu(x_1)+\sigma\delta} q(\boldsymbol{u})[\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) - \nabla_{\boldsymbol{u}}\phi(\boldsymbol{u})]K_\alpha(\boldsymbol{x}-\boldsymbol{u})\left(1, \frac{\partial\mu}{\partial x_1}\right)\mathrm{d}x_1\right]\mathrm{d}\boldsymbol{u} \\
=: \int\limits_{\boldsymbol{u}\in II} q(\boldsymbol{u})G(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2 \\
= \int\limits_{\boldsymbol{u}\in II} q(\boldsymbol{u})[G(u_1,u_2) - G(u_1,\mu(u_1))]\mathrm{d}u_1\mathrm{d}u_2,
\end{aligned} \tag{2.30}$$

where (2.30) follows from Remark 2.12. Noting that by $K_\alpha \in L^1$:

$$|G(u_1,u_2) - G(u_1,\mu(u_1))| \leq C\delta$$

for some constant C independent of δ, and that the domain is $O(\delta)$, this gives (B1) is $O(\delta^2)$ as required. Next we define:

$$Q(\boldsymbol{x}) = \int\limits_{\boldsymbol{u}\in II} K_\alpha(\boldsymbol{x}-\boldsymbol{u})[\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) - \nabla_{\boldsymbol{u}}\phi(\boldsymbol{u})]q(\boldsymbol{u})\mathrm{d}\boldsymbol{u} \tag{2.31}$$

and write:

$$(\mathrm{B2}) = \int\limits_{\boldsymbol{x}\in II} Q(\boldsymbol{x})\nabla^\perp_{\boldsymbol{x}} q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{2.32}$$

Using $K_\alpha \in L^1$ we can show that $\nabla^2_{x'}Q$ is bounded. By an application of Corollary 2.14 on (2.32) we have:

$$(\mathrm{B2}) = O(\delta^2) - \int\limits_{x_2=\mu(x_1)} Q(\boldsymbol{x})\Big(1, \frac{\partial \mu}{\partial x_1}(x_1,t)\Big)\mathrm{d}x_1$$

$$O(\delta^2) - \int\limits_{x_2=\mu(x_1)} \int\limits_{\boldsymbol{u}\in II} K_\alpha(\boldsymbol{x}-\boldsymbol{u})[\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) - \nabla_{\boldsymbol{u}}\phi(\boldsymbol{u})]q(\boldsymbol{u})\Big(1, \frac{\partial \mu}{\partial x_1}(x_1,t)\Big)\mathrm{d}\boldsymbol{u}\mathrm{d}x_1$$

$$= O(\delta^2) - \int\limits_{\boldsymbol{u}\in II} q(\boldsymbol{u}) \int\limits_{x_2=\mu(x_1)} K_\alpha(\boldsymbol{x}-\boldsymbol{u})[\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x}) - \nabla_{\boldsymbol{u}}\phi(\boldsymbol{u})]\Big(1, \frac{\partial \mu}{\partial x_1}(x_1,t)\Big)\mathrm{d}x_1\mathrm{d}\boldsymbol{u} \tag{2.33}$$

$$= O(\delta^2) + \int\limits_{\boldsymbol{u}\in II} q(\boldsymbol{u})P(\boldsymbol{u})\mathrm{d}\boldsymbol{u}, \tag{2.34}$$

where $P(\boldsymbol{u})$ is defined by the inner integral in (2.33). Writing:

$$(2.34) = O(\delta^2) + \int\limits_{\boldsymbol{u}\in II} q(\boldsymbol{u})[P(u_1,\mu(u_1)) - P(u_1,u_2)]\mathrm{d}\boldsymbol{u} \tag{2.35}$$

using the spine condition in Remark 2.12. Using the same argument as for (B1) on the integral term in (2.35), we obtain that (B2) is precisely $O(\delta^2)$ as required. For the final term:

$$(\mathrm{B3}) = \frac{1}{4}\sum_{\sigma_1,\sigma_2=\pm 1} \int\limits_{x_2=\mu(x_1)+\sigma_1\delta} \int\limits_{u_2=\mu(u_1)+\sigma_2\delta} \phi(x_1,x_2)\Big(\frac{\partial \mu}{\partial u_1} - \frac{\partial \mu}{\partial x_1}\Big)K_\alpha(\boldsymbol{x}-\boldsymbol{u})\mathrm{d}x_1\mathrm{d}u_1. \tag{2.36}$$

For $\sigma_1 = \sigma_2$:

$$\frac{1}{4}\sum_{\sigma=\pm 1} \int\limits_{x_2=\mu(x_1)+\sigma\delta} \int\limits_{u_2=\mu(u_1)+\sigma\delta} \phi(x_1,x_2)\Big(\frac{\partial \mu}{\partial u_1} - \frac{\partial \mu}{\partial x_1}\Big)K_\alpha(\boldsymbol{x}-\boldsymbol{u})\mathrm{d}x_1\mathrm{d}u_1$$

$$= \frac{1}{2} \int\limits_{x_2=\mu(x_1)} \int\limits_{u_2=\mu(u_1)} \phi(x_1,x_2)\Big(\frac{\partial \mu}{\partial u_1} - \frac{\partial \mu}{\partial x_1}\Big)K_\alpha(x_1-u_1,\mu(x_1)-\mu(u_1))\mathrm{d}x_1\mathrm{d}u_1 + O(\delta^2) \tag{2.37}$$

using a Taylor expansion, and for the remaining terms we have:

$$\frac{1}{4}\sum_{\sigma=\pm 1}\int\limits_{x_2=\mu(x_1)+\sigma\delta}\int\limits_{u_2=\mu(u_1)-\sigma\delta}\phi(x_1,x_2)\left(\frac{\partial\mu}{\partial u_1}-\frac{\partial\mu}{\partial x_1}\right)K_\alpha(\boldsymbol{x}-\boldsymbol{u})\mathrm{d}x_1\mathrm{d}u_1=F(\delta). \tag{2.38}$$

Noting that $F(\delta)=F(0)+O(\delta^2)$ follows on showing $F'(\delta)\leq C\delta$; the proof is the same as in Fefferman et al. (2012) up to the change in singularity of the kernel and so we omit the lengthy calculations. Combining the estimates on (B1)-(B3) completes the proof. □

2.5 Discussion

When studying the evolution of sharp fronts for the SQG equation, the analogous problem for three-dimensional Euler is the evolution of a vortex line (as discussed in the Introduction). The derivation of the sharp front equation uses tools not available for the Euler case. Current study involves the study of almost-sharp fronts and the limiting procedure as the thickness of the front δ approaches 0 as an insight into this problem (Córdoba et al., 2004).

Within this Chapter, we have summarised several results that have been proven for the SQG equation, regarding estimates on almost-sharp front solutions in the limit as $\delta\to 0$. For this case $\alpha=1$, the existence of almost-sharp front solutions remains an open question; the most important case being the existence of smooth solutions of such a type. In particular, solutions that exist for time independent of δ. The construction of such solutions is studied in Fefferman and Rodrigo (2012).

We study the analogous problem for the α-equation; that is we attempt to construct almost-sharp front solutions. In studying almost-sharp fronts for this system, we attempt to introduce several methods which could be applied to the SQG case, with the study of this system being simpler (as previously discussed). The most ideal result would be to show that there exist smooth almost-sharp front solutions to (2.1)-(2.2) and study these in the limit as $\alpha\to 1$. This would also allow us to utilise the results proved within this chapter and to see which estimates we may recover in the limit. However this also remains an open question (see Chapter 3).

With the results presented in §2.3 and §2.4 assuming the existence of almost-sharp front solutions, the focus of this thesis is now on the study of their construction. We study the smooth case in the next chapter and discuss the open problems that remain following the result presented here. The analytic case then forms the rest of the thesis in which we are able to prove existence results for approximate and exact solutions to the α-equation; the final result giving solutions taking the form of an almost-sharp front. All of these results give the time of existence independent of the thickness of the front.

Chapter 3

Limit Equations in the Smooth Case

The results described in the previous chapter (the estimates in both §2.3 and §2.4) assume the existence of almost-sharp fronts for the α-equation; the focus of this chapter is the study of their construction. We define below a family of almost-sharp fronts, indexed by a parameter $\delta > 0$ relating to the thickness of the front (as seen in Chapter 2). For this specific family, we derive the limit equations as δ approaches 0 for each fixed value of α and prove an approximation result for smooth solutions. The study of analytic solutions forms the basis of the next chapter. We consider the case only when $0 < \alpha < 1$; the approximation result stated in §3.2 is an analogue of the result proved in Fefferman and Rodrigo (2012) for the SQG equations ($\alpha = 1$).

Recall that the α-equation is given by:

$$\partial_t q + u \cdot \nabla q = 0 \tag{3.1}$$

where:

$$u = (-\Delta)^{-(2-\alpha)/2} \nabla^{\perp} q. \tag{3.2}$$

We continue to study this system when posed on a two-dimensional spatial domain $(x, y) \in \mathbb{R}/\pi\mathbb{Z} \times \mathbb{R}$ with $t \in [0, T]$, that is we are considering periodic behaviour in the horizontal direction. Any functions will be defined for $(x, y) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}$ and extended periodically to the whole plane.

The concept of sharp fronts has been discussed in previous chapters and the sharp-front equations derived in §2.2. Almost-sharp fronts were introduced in Chapter 2, and a detailed definition of these was given in §2.4; roughly speaking, these are regularisations of a sharp front in which the solution changes smoothly between two constant values[1] in a small strip along the boundary. We aim to construct a family of almost-sharp fronts - weak solutions of the α-equation, of the form:

$$q(x,y,t) = \begin{cases} \frac{1}{2} & \text{if} \quad y \geq \varphi(x,t) + \delta \\ \text{bounded} & \text{if} \quad |y - \varphi(x,t)| < \delta \\ \frac{1}{2} & \text{if} \quad y \leq \varphi(x,t) - \delta \end{cases} \tag{3.3}$$

where, as previously, the given function $\varphi(x,t)$ is periodic in the x variable, and $\delta > 0$ acts as a parameter for our family of almost-sharp fronts. During the construction it will be shown that it is necessary for the curve $y = \varphi(x,t)$ to satisfy the sharp front equation (2.10).

3.1 Change of Coordinates

In order to study almost-sharp fronts of the form (3.3), we first need to introduce a smooth change of coordinates, enabling us to study the evolution on a fixed domain of size independent of δ.

The new coordinates describe a neighbourhood of the curve $y = \varphi(x,t)$ using renormalized arc length, s, and a distance to the curve, ξ, which will be scaled by the parameter δ. This coordinate system was introduced in the study of the construction of almost sharp-fronts for the SQG equation (when $\alpha = 1$), in Fefferman and Rodrigo (2012). The change of variables remain the same for the case $\alpha < 1$ and this section contains an outline of the required construction as presented in that paper. Appendix B contain full details of the calculations for the results given below.

The renormalised arc length for the curve $\varphi(x,t)$ of period π in the x-direction is defined to be:

$$R(x,t) = \frac{1}{L(t)} \int_{-\frac{\pi}{2}}^{x} (1 + (\varphi'^2(\bar{x},t)))^{1/2} \mathrm{d}\bar{x} \tag{3.4}$$

where $L(t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + (\varphi'^2(\bar{x},t)))^{1/2} \mathrm{d}\bar{x}$ represents the length of the curve. We use prime notation to denote the derivative with respect to the first variable. When considered only as a function of x, R is invertible, and so we use R^{-1} to construct a map between the fixed domain $(s,\xi) \in [0,1) \times [-1,1]$ and the corresponding δ-

[1] For consistency we continue to choose these values to be $-\frac{1}{2}$ and $\frac{1}{2}$.

neighbourhood of the curve φ as defined in (3.3), and illustrated in Figure 3.1. This map is given by:

$$(x, y) = (R^{-1}(s, t), \varphi(R^{-1}(s, t), t)) + \mathbf{n}(R^{-1}(s, t))\xi\delta \tag{3.5}$$

where $\mathbf{n}(R^{-1}(s, t))$ is the unit normal to the curve φ at the point $R^{-1}(s, t)$, and $\mathbf{t}(R^{-1}(s, t)$ the corresponding unit tangent vector:

$$\mathbf{n}(R^{-1}(s, t)) = \frac{(-\varphi'((R^{-1}(s, t)), t), 1)}{\|(-\varphi'((R^{-1}(s, t)), t), 1)\|},$$
$$\mathbf{t}(R^{-1}(s, t)) = \frac{(1, \varphi'((R^{-1}(s, t)), t))}{\|(1, \varphi'((R^{-1}(s, t)), t))\|}.$$

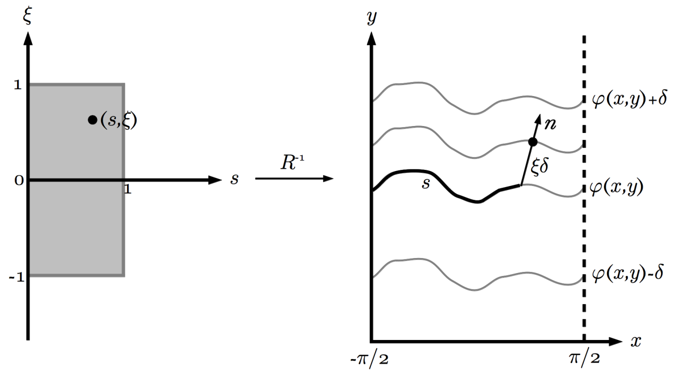

Figure 3.1: Change of Coordinates

By the definition of the almost-sharp front that was introduced in (3.3), the parameter δ corresponds to half of the thickness of the front. Since φ is given, it is clear that there exists a value δ_0, depending on the curvature of φ, such that for $\delta \leq \delta_0$ the map defined in (3.5) is injective. At this point we also introduce a new time variable τ and now consider, in the new variables, a family of solutions to the α-equation of the form:

$$q(x, y, t) = \Omega(s, \xi, \tau) \tag{3.6}$$

where, by (3.5) Ω satisfies:

$$\Omega(s,\xi,\tau)=\begin{cases}\frac{1}{2} & \xi\geq 1\\ \text{smooth} & |\xi|<1\\ -\frac{1}{2} & \xi\leq -1\end{cases} \tag{3.7}$$

Remark 3.1. *For the same reasoning as in Chapter 2, the constant values of* Ω *have been chosen in such a way that we get some cancellations, which simplify the results that follow. In this case we have that* $\int_1^1 \Omega_\xi d\xi = 1$*, and* $\xi\Omega(\xi,\tau)|_{-1}^{1} = 0$*.*

The fixed domain as constructed is of size independent of δ and so the family of solutions defined in (3.7) lose some of their dependence on δ, and their behaviour is somewhat controlled. In fact, it is clear from the construction that Ω will be smooth in all variables; the derivatives will still depend on δ which will be seen shortly. In order to study the limit equations we will assume that the Sobolev norms of Ω with respect to the variables s and ξ, while dependent on δ, are uniformly bounded for all $\delta\leq\delta_0$. This is not known to be true in the smooth case - in Chapter 6 we obtain a construction of a family of almost-sharp fronts which satisfy this assumption. In Fefferman and Rodrigo (2012) it is remarked that when $\alpha=1$ the dependence of Ω on δ is bad, in such a way that the τ-derivative of Ω is logarithmic in δ. We show that for $0<\alpha<1$ the singularities concerned mean that we do not encounter this logarithmic behaviour, and obtain a much simpler form for the limit equation.

We now write the α-equation in terms of $\Omega(s,\xi,\tau)$ as defined above, where we have:

$$(x,y,t)=(R^{-1}(s,\tau),\varphi(R^{-1}(s,\tau),\tau),\tau)+\left(\frac{(-\varphi'((R^{-1}(s,\tau)),\tau),1)}{\|(-\varphi'((R^{-1}(s,\tau)),\tau),1)\|}\xi\delta,0\right) \tag{3.8}$$

and simplify some of the terms that appear by writing $\varphi(s)=\varphi(R^{-1}(s,\tau))$, $\varphi'(s)=\varphi'(R^{-1}(s,\tau))$ and $\varphi''(s)=\varphi''(R^{-1}(s,\tau))$. When it is clear, we will suppress some of the arguments. We first have that:

$$\partial_x=\frac{1}{Det(s)}\frac{\partial y}{\partial\xi}\partial_s-\frac{1}{Det(s)}\frac{\partial y}{\partial s}\partial_\xi,\quad \partial_y=\frac{1}{Det(s)}\frac{\partial x}{\partial s}\partial_\xi-\frac{1}{Det(s)}\frac{\partial x}{\partial\xi}\partial_s,$$

$$\partial_t=\frac{I}{Det(s)}\partial_s+\frac{II}{Det(s)}\partial_\xi+\partial_\tau,$$

where:

$$Det(s) = \frac{\partial x}{\partial s}\frac{\partial y}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial s}, \quad I = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \xi}, \quad II = \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial \tau}$$

with:

$$\frac{\partial x}{\partial s} = R_s^{-1} + \left[\frac{-\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\right]\xi\delta, \quad \frac{\partial y}{\partial s} = \varphi' R_s^{-1} - \frac{\varphi'\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\xi\delta,$$

$$\frac{\partial x}{\partial \xi} = \frac{-\varphi'\delta}{(1+\varphi'^2(s))^{1/2}}, \quad \frac{\partial y}{\partial \xi} = \frac{\delta}{(1+\varphi'^2(s))^{1/2}},$$

$$\frac{\partial x}{\partial \tau} = R_\tau^{-1} + \left[\frac{-\varphi'' R_\tau^{-1} - \varphi'_\tau}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2(\varphi'' R_\tau^{-1} + \varphi'_\tau)}{(1+\varphi'^2(s))^{3/2}}\right]\xi\delta,$$

$$\frac{\partial y}{\partial \tau} = \varphi' R_\tau^{-1} + \varphi_\tau - \frac{\varphi'(\varphi'' R_\tau^{-1} + \varphi'_\tau)}{(1+\varphi'^2(s))^{3/2}}\xi\delta.$$

Using the inverse function theorem to determine that $R_s^{-1} = \frac{L}{(1+\varphi'^2(s))^{1/2}}$, we find the following simplified terms:

$$Det(s) = L\delta - L\frac{\varphi''(s)}{(1+\varphi'^2(s))^{3/2}}\xi\delta^2, \tag{3.9}$$

$$I = \frac{\delta}{(1+\varphi'^2(s))^{1/2}}\left[-(1+\varphi'^2(s))R_\tau^{-1} - \varphi'(s)\varphi_\tau(s) + \frac{\varphi''(s)R_\tau^{-1} + \varphi'_\tau(s)}{(1+\varphi'^2(s))^{1/2}}\xi\delta\right],$$

$$II = -\frac{L\varphi_\tau(s)}{(1+\varphi'^2(s))^{1/2}} + \frac{L\varphi''(s)\varphi_\tau(s)}{(1+\varphi'^2(s))^2}\xi\delta,$$

and a series expansion provides the following estimates:

$$\frac{1}{Det(s)} = \frac{1}{L\delta} - \frac{1}{L}\frac{\varphi''(s)}{(1+\varphi'^2(s))^{3/2}}\xi + O(\delta), \quad \frac{\delta}{Det(s)} = \frac{1}{L} + O(\delta). \tag{3.10}$$

We find the following expressions for the space and time derivatives written in the new variables, with error terms highlighted as required for the derivation of the limit equation in §3.4.

$$\partial_t q = \Omega_\tau(s,\xi,\tau) + \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}\left[-(1+\varphi'^2(s))R_\tau^{-1} - \varphi'\varphi_\tau\right]\Omega_s(s,\xi,\tau)$$
$$-\frac{1}{\delta}\frac{\varphi_\tau}{(1+\varphi'(s))^{1/2}}\Omega_\xi(s,\xi,\tau) + 2\frac{\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2}\xi\Omega_\xi(s,\xi,\tau) + O(\delta). \tag{3.11}$$

$$\nabla q = \mathbf{t}(s)\frac{\delta}{Det(s)}\Omega_s(s,\xi,\tau) + \mathbf{n}(s)\frac{L}{Det(s)}\Omega_\xi(s,\xi,\tau)$$
$$-\mathbf{n}(s)\frac{\delta\xi}{Det(s)}\frac{\varphi'' L}{(1+\varphi'^2)^{3/2}}\Omega_\xi(s,\xi,\tau). \tag{3.12}$$

$$\nabla^\perp q = \mathbf{n}(s)\frac{\delta}{Det(s)}\Omega_s(s,\xi,\tau) - \mathbf{t}(s)\frac{L}{Det(s)}\Omega_\xi(s,\xi,\tau)$$
$$+\mathbf{t}(s)\frac{\delta\xi}{Det(s)}\frac{\varphi'' L}{(1+\varphi'^2)^{3/2}}\Omega_\xi(s,\xi,\tau). \tag{3.13}$$

We now study the term $u \cdot \nabla q$, where u is as defined in (3.2) and, for the smooth case, can be written as a convolution with the kernel K_α as detailed in §2.1. We aim to derive the limit equation in this case; we show that on writing $u \cdot \nabla q$ in the new coordinate system, some of the terms that arise will be error terms and we can simplify many of the terms. The derivation of the limit equation relies on an adapted lemma from Fefferman and Rodrigo (2012) outlined in §3.3.

Let K_α^δ denote the kernel in the new coordinates which will be defined when needed. Under the change of coordinates as outlined we have:

$$u(s,\xi,\tau) = \iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} K_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})\nabla^\perp q(\bar{s},\bar{\xi})Det(\bar{s})\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}, \tag{3.14}$$

where we have highlighted the dependence of the kernel on δ.

Remark 3.2. *Note that the unit normal and tangent vectors,* $\mathbf{n}$ *and* $\mathbf{t}$*, depend on* s*. The contributions from such terms in* u *and* ∇q *are therefore different; see (B.17) - (B.20).*

Utilising the previous calculations we have that:

$$u \cdot \nabla q =$$

$$-\frac{L^2}{Det(s)} \iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_\xi \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{3.15}$$

$$+\frac{L\delta}{Det(s)} \iint K_\alpha^\delta \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \left[\Omega_\xi \Omega_{\bar{s}} - \Omega_s \Omega_{\bar{\xi}}\right] \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{3.16}$$

$$\begin{aligned} &+\frac{L^2\delta}{Det(s)} \iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \left(\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} \right. \\ &\qquad \left. + \frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}} \right) \Omega_\xi \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \end{aligned} \tag{3.17}$$

$$-\frac{\delta^2}{Det(s)} \iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_s \Omega_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{3.18}$$

$$\begin{aligned} &+\frac{L\delta^2}{Det(s)} \iint K_\alpha^\delta \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \left(\frac{\bar{\xi}\Omega_s \Omega_{\bar{\xi}} \varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}} \right. \\ &\qquad \left. + \frac{\xi \Omega_\xi \Omega_{\bar{s}} \varphi''(s)}{(1+\varphi'^2(s))^{3/2}} \right) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \end{aligned} \tag{3.19}$$

$$\begin{aligned} &-\frac{L^2\delta^2}{Det(s)} \iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}} \\ &\qquad \times \frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} \Omega_\xi \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \end{aligned} \tag{3.20}$$

where all double integrals are taken over the domain $\mathbb{R}/\mathbb{Z} \times [-1,1]$. Note that the restriction of the integral to $\xi \in [-1,1]$ is the case by the definition of the family of almost-sharp fronts in (3.7). Outside of this region the derivatives $\Omega_{\bar{s}}$ and $\Omega_{\bar{\xi}}$ are identically 0.

Remark 3.3. *In (3.11), the equation for the time derivative in the new variables, the third term is $O\left(\frac{1}{\delta}\right)$ and could cause a problem with blow up in taking the limit as δ approaches 0. We show that this term causes no such problem and does not appear in the limit equation; in particular we show on rearranging (3.15) that this term cancels due to matching of coefficients and the sharp front equation stated in (2.10). For (3.11), we notice that the coefficient of Ω_s is of order one (with respect to δ).*

Remark 3.4. *Terms of the form $\xi\Omega_\xi$ that appear for example as the fourth term in (3.11) and, as we will see in §3.4, arise from some of the integrals (3.15)-(3.20), do not pose a problem in the smooth case. We only consider $\xi \in [-1,1]$ and so this term will always remain bounded. For the analytic case, the appearance of this term would make subsequent analysis more complicated as ξ takes all values in $\mathbb{R}$.*

In fact, we are able to show that under a different change of coordinates, this term no longer appears in the corresponding limit equation (see Chapter 4).

We have already shown that $\frac{1}{Det(s)}$ is $O\left(\frac{1}{\delta}\right)$ (3.10), and as covered in Chapter 2, we have that for $K_\alpha \in L^1$, the integrals in (3.15)-(3.20) are all finite. We use these facts to briefly outline some estimates of the terms that arise from $u \cdot \nabla q$. The terms (3.18)-(3.20) are automatically error terms (with respect to δ) and so will need no further analysis when deriving the limit equation. The integral in (3.15) has a coefficient of order $\frac{1}{\delta}$; by further study of this term we can show that this is equivalent to a sharp front term and some error term, and so no terms of this order will appear in the final equations (see Remark 3.3). The remaining terms (3.16) and (3.17) are both $O(1)$ and to study these further we will require some technical results which will be introduced in §3.3.

Before giving the statements of the approximation results we complete the set-up required for the derivation of the limit equations by now giving the form of the sharp front equation (originally derived in Theorem 2.5), under this coordinate transformation. Note that we use the simplified version of the first kernel (see Remark 2.1). As we work in the periodic case, we can also simplify the presentation by taking $\chi = \chi(x - \bar{x})$ (Fefferman and Rodrigo, 2012) . We have then, for the evolution of our periodic function $\varphi(s, \tau)$, the sharp-front equation:

$$\frac{\partial \varphi}{\partial \tau}(s, \tau) = \int_{\mathbb{R}/\mathbb{Z}} \frac{(\varphi'(s) - \varphi'(\bar{s}))\chi(R^{-1}(s) - R^{-1}(\bar{s}))}{[(R^{-1}(s) - R^{-1}(\bar{s}))^2 + (\varphi(s) - \varphi(\bar{s}))^2]^{\alpha/2}} \frac{L}{(1 + \varphi'^2(\bar{s}))^{1/2}} \mathrm{d}\bar{s}. \quad (3.21)$$

3.2 Statement of the Theorem

We now introduce the main result of this chapter, the proof of which will be carried out in the remaining sections. As previously stated, this result is an analogue for the case when $\alpha = 1$ as given in Fefferman and Rodrigo (2012); we first give an overview of this result, and a comparison to the presentation in the case when $0 < \alpha < 1$.

In Fefferman and Rodrigo (2012) the authors define an approximate solution as follows:

Definition 3.5. *An approximate solution of SQG is defined to be a family of functions $\Omega(s, \xi, \tau)$, parametrized by δ (as defined previously), such that the resulting $q(x, y, t)$ (also parametrised by δ), satisfies (1.5) and (1.7) with the right hand side of (1.5) replaced by an error $o(1)$ as $\delta \to 0$.*

We extend this definition to the case when $0 < \alpha < 1$ by simply considering instead the right hand side of (3.1).

For the SQG equations, it is initially not possible to take a formal limit of the equations under the new coordinates due to the appearance of logarithmic terms that arise as a result of the singularity of the kernel. For this purpose the authors first derive an equation for the function $h(s) = \int \Omega(s,\xi)\mathrm{d}\xi$, by integrating the equation for Ω with respect to the variable ξ. Such an equation (the h-equation), has a formal limit and the following result can be proved, considering h as a known function:

Theorem 3.6. *(Fefferman and Rodrigo, 2012) Given a curve $y = \varphi(x,t)$ satisfying the sharp front equation, consider a family of functions $\Omega(s,\xi,\tau)$ (indexed by δ) based on that curve via the change of coordinates*

$$(x,y,t) = (R^{-1}(s,\tau), \varphi(R^{-1}(s,\tau),\tau),\tau) + \frac{(-\varphi'(R^{-1}(s,\tau),\tau),1,0)}{\|(-\varphi'(R^{-1}(s,\tau),\tau),1,0\|}\xi\delta \tag{3.22}$$

such that for each fixed τ

- *Ω is a smooth function of s and ξ with Sobolev norms bounded independently of δ,*
- *$\Omega(s,\xi,0) = \Omega_0$,*

then Ω is an approximate solution of SGQ if and only if it solves equation

$$\begin{aligned} o(1) &= \Omega_\tau + A_1\Omega_\xi + A_2\Omega_s \\ &+ \log(\delta)\frac{2}{L}\left[\Omega_\xi(s,\xi)h_{\bar{s}}(s) - \Omega_s(s,\xi)\right] \\ &+ \frac{2}{L}\Omega_\xi(s,\xi)\int \Omega_{\bar{s}}(s,\bar{\xi})\log(|\bar{\xi}-\xi|)\mathrm{d}\bar{\xi} - \frac{2}{L}\Omega_s(s,\xi)\int \Omega_{\bar{\xi}}(s,\bar{\xi})\log(|\bar{\xi}-\xi|)\mathrm{d}\bar{\xi} \end{aligned} \tag{3.23}$$

where the functions A_i can be explicitly computed and depend only on φ and h, where h is a solution of the h-equation with $h_0(x,t) = \int \Omega_0(x,\bar{\xi},t)\mathrm{d}\bar{\xi}$.

When $\alpha < 1$ we will see that such logarithmic behaviour does not occur, and so on rewriting the equations in the new coordinates and finding estimates on the terms that arise, we are able to take the formal limit and obtain:

Theorem 3.7. *Given a curve $y = \varphi(x,t)$ satisfying the sharp front equation (equation (2.10) with $\eta \equiv 0$), and a family of functions $\Omega(s,\xi,\tau)$ (indexed by $\delta > 0$) defined by that curve φ via the change of coordinates*

$$(x,y,t) = (R^{-1}(s,\tau), \varphi(R^{-1}(s,\tau),\tau),\tau) + \frac{(-\varphi'(R^{-1}(s,\tau),\tau),1,0)}{\|(-\varphi'(R^{-1}(s,\tau),\tau),1,0)\|}\xi\delta \tag{3.24}$$

such that for each τ we have:

- *Ω is a smooth function of s and ξ with Sobolev norms bounded independently of δ,*
- *$\Omega(s,\xi,0) = \Omega_0$.*

Then Ω is an approximate solution of the α-equation if and only if it satisfies the equation

$$\begin{aligned} o(1) = \Omega_\tau + A_1\Omega_s + A_2\xi\Omega_\xi + B_1 \iint T_1\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_\xi \\ B_2 \iint T_2\Omega_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_\xi + B_3 \iint T_3\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_s \end{aligned} \tag{3.25}$$

where the functions A_i, B_i and T_i can be explicitly computed.

An ideal theorem to prove in the case $\alpha < 1$ would be existence of solutions in short-time to such a limit equation. We see that this is precisely the case when we consider analytic solutions in the next chapter. For the smooth case we are, at present, only able to derive the limit equation and prove the approximation result.

3.3 Preliminary Lemmas

In order to give the first derivation of the limit equation, introduced here are several results that are adapted from the case when $\alpha = 1$, given in Fefferman and Rodrigo (2012), providing estimates that can be used to determine the behaviour in the limit of the highlighted terms from $u \cdot \nabla q$. All results below hold for values $0 < \alpha < 1$.

Lemma 3.8. *Let a and g be smooth periodic functions of period π. Assume that g is non-negative and has a non-degenerate local minimum at 0 (that is $g'(0) = 0$ and $g''(0) > 0$), and that $\frac{g(x)}{|sin(x)|^2} > c$ and $g(0) = 0$. Then there exists c' such that for $0 < \mu < c'$ we have*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a(y)}{(g(y)+\mu^2)^{\alpha/2}}\mathrm{d}y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a(y)}{(g(y))^{\alpha/2}}\mathrm{d}y + a(0)\left(\frac{1}{2}g''(0)\right)^{-1/2} O(\mu^{1-\alpha}) + Error \tag{3.26}$$

where $|Error| \leq c''\mu^2$. The constants c' and c'' depend only on the constants c and bounds on both a and g.

Proof By the assumptions on the function g, we first split the integral into an inner and outer region. The inner region is a small interval around the origin, whose size is determined by the the behaviour of g. On this region we have additional information about the form of this function by the assumptions on the local minimum. On the outer region, that is the complement of the above, we use that g is a bounded function. We study the following:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a(y)}{(g(y)+\mu^2)^{\alpha/2}}\mathrm{d}y = \int_{y_{lo}}^{y_{hi}} \frac{a(y)}{(g(y)+\mu^2)^{\alpha/2}}\mathrm{d}y + \int_{[-\frac{\pi}{2},\frac{\pi}{2}]/[y_{lo},y_{hi}]} \frac{a(y)}{(g(y)+\mu^2)^{\alpha/2}}\mathrm{d}y \quad (3.27)$$

where the limits y_{lo} and y_{hi} will be determined by g. Near to the origin $y = 0$, we make a change of variables: $g(y) = z^2$ on some neighbourhood $z \in [-z_0, z_0]$ such that the endpoints are defined by $g(y_{lo}) = g(y_{hi}) = z_0^2$.

We also have that:

$$a(y) = A((g(y))^{1/2})\frac{\mathrm{d}}{\mathrm{d}y}[(g(y))^{1/2}] \quad (3.28)$$

where the function $A(z)$ satisfies:

$$a(y)\mathrm{d}y = A(z)\mathrm{d}z \quad \text{for } z \in [-z_0, z_0].$$

Using a Taylor expansion for g, since this function had a local minimum at 0, and given $g(0) = 0$, in a small neighbourhood of the origin we have $g(y) = \frac{1}{2}g''(0)y^2 + O(y^3)$, and the required form: $(g(y))^{1/2} = (\frac{1}{2}g''(0))^{1/2} + O(y^2)$. This gives the following convergence:

$$\frac{\mathrm{d}}{\mathrm{d}y}(g(y))^{1/2} \to \left(\frac{1}{2}g''(0)\right)^{1/2} \quad \text{as } y \to 0^+.$$

Substituting this into (3.28) we see that

$$A(0) = a(0)\left(\frac{1}{2}g''(0)\right)^{-1/2}. \quad (3.29)$$

By this change of variables, we rewrite the inner part of (3.27) as follows:

$$\int_{y_{lo}}^{y_{hi}} \frac{a(y)}{(g(y)+\mu^2)^{\alpha/2}} \mathrm{d}y = \int_{-z_0}^{z_0} \frac{A(z)}{(z^2+\mu^2)^{\alpha/2}} \mathrm{d}z \tag{3.30}$$

$$= A(0) \int_{-z_0}^{z_0} \frac{\mathrm{d}z}{(z^2+\mu^2)^{\alpha/2}} + \int_{-z_0}^{z_0} \frac{A(z)-A(0)}{|z|^\alpha} \mathrm{d}z \tag{3.31}$$

$$+ \int_{-z_0}^{z_0} (A(z)-A(0)-A'(0)z) \left[\frac{1}{(z^2+\mu^2)^{\alpha/2}} - \frac{1}{|z|^\alpha} \right] \mathrm{d}z. \tag{3.32}$$

We first study the term (3.32). To do this we split the integral further; for $|z| > 2\mu$ we have, using a binomial expansion (valid as $|\frac{\mu}{z}| < \frac{1}{2}$):

$$\left| \frac{1}{(z^2+\mu^2)^{\alpha/2}} - \frac{1}{|z|^\alpha} \right| = \frac{1}{|z|^\alpha} \left| \left(1 + \frac{\mu^2}{z^2}\right)^{\alpha/2} - 1 \right| \leq C_1 \frac{\mu^2}{|z|^{2+\alpha}}$$

where the constant C_1 depends on α. If $|z| \leq 2\mu$ we have:

$$\left| \frac{1}{(z^2+\mu^2)^{\alpha/2}} - \frac{1}{|z|^\alpha} \right| \leq C_2 \frac{1}{|z|^\alpha}$$

for some constant C_2. By a Taylor expansion $|A(z) - A(0) - A'(0)z| \leq C_3 z^2$ in a small neighbourhood of the origin, and so

$$\begin{aligned} &\int_{-z_0}^{z_0} (A(z)-A(0)-A'(0)z) \left[\frac{1}{(z^2+\mu^2)^{\alpha/2}} - \frac{1}{|z|^\alpha} \right] \mathrm{d}z \\ &\quad \lesssim \int_{|z|\leq 2\mu} z^2 \frac{1}{|z|^\alpha} \mathrm{d}z + \int_{2\mu<|z|<z_0} z^2 \frac{\mu^2}{|z|^{2+\alpha}} \mathrm{d}z \\ &\quad \lesssim \mu^{3-\alpha} + \mu^2 + \mu^{3-\alpha}. \end{aligned} \tag{3.33}$$

Taking leading terms we see that (3.32) is $O(\mu^2)$. We use the same method to analyse the following term from (3.31):

$$A(0) \int_{-z_0}^{z_0} \frac{1}{(z^2+\mu^2)^{\alpha/2}} - \frac{1}{|z|^\alpha} dz.$$

Considering the two regions as defined before, that is $|z| > 2\mu$ and $|z| \le 2\mu$, we find that:

$$A(0)\int_{-z_0}^{z_0} \frac{1}{(z^2+\mu^2)^{\alpha/2}} - \frac{1}{|z|^\alpha} dz \lesssim A(0)\int_{|z|\le 2\mu} \frac{1}{|z|^\alpha} dz + A(0)\int_{2\mu<|z|<z_0} \frac{\mu^2}{|z|^{2+\alpha}} dz$$
$$\lesssim A(0)\mu^{1-\alpha} + A(0)\mu^2 + A(0)\mu^{1-\alpha} \tag{3.34}$$

and so this term is of the form $A(0)O(\mu^{1-\alpha})$. Recall that:

$$\int_{-z_0}^{z_0} \frac{A(z)}{|z|^\alpha} dz = \int_{y_{lo}}^{y_{hi}} \frac{a(y)}{(g(y))^{\alpha/2}} dy.$$

We have shown for the inner part that:

$$\int_{y_{lo}}^{y_{hi}} \frac{a(y)}{(g(y)+\mu^2)^{\alpha/2}} dy = \int_{y_{lo}}^{y_{hi}} \frac{a(y)}{(g(y))^{\alpha/2}} dy + A(0)O(\mu^{1-\alpha}) + O(\mu^2). \tag{3.35}$$

It remains to calculate the contribution from the outer part. To do this we use that g is bounded on this domain. Using the same technique from before of estimating the difference $\left|\frac{1}{(g(y)+\mu^2)^{\alpha/2}} - \frac{1}{(g(y))^{\alpha/2}}\right|$, this time with an lower bound on g.

Setting $F(\eta) = \frac{1}{(g(y)+\mu^2)^{\alpha/2}}$ so that $F'(\eta) = -\frac{\alpha}{2}\frac{1}{(g(y)+\mu^2)^{(\alpha+2)/2}}$, we have by the mean value theorem and taking $\eta = \mu^2$:

$$\left|\int_{y_{hi}}^{\frac{\pi}{2}} \frac{a(y)}{(g(y)+\mu^2)^{\frac{\alpha}{2}}} - \frac{a(y)}{(g(y))^{\frac{\alpha}{2}}} dy\right| \le \int_{y_{hi}}^{\frac{\pi}{2}} \left|\frac{a(y)}{(g(y)+\mu^2)^{\frac{\alpha}{2}}} - \frac{a(y)}{(g(y))^{\frac{\alpha}{2}}}\right| dy$$
$$\le \int_{y_{hi}}^{\frac{\pi}{2}} \frac{\alpha}{2} \sup_{\tilde{\eta}\in(0,\mu^2)} \left|\frac{1}{(g(y)+\tilde{\eta})^{(\alpha+2)/2}}\right| \mu^2 \mathrm{d}y$$
$$\le \int_{y_{hi}}^{\frac{\pi}{2}} \frac{\alpha}{2} \sup_{\tilde{\eta}\in(0,\mu^2)} \left|\frac{1}{(g(y))^{(\alpha+2)/2}}\right| \mu^2 \mathrm{d}y \tag{3.36}$$

which by the lower bound on g, and an analogous result for second outer part, we have that:

$$\int_{y_{hi}}^{\frac{\pi}{2}} \frac{a(y)}{(g(y)+\mu^2)^{\frac{\alpha}{2}}} - \frac{a(y)}{(g(y))^{\frac{\alpha}{2}}} dy = O(\mu^2), \tag{3.37}$$

$$\int_{-\frac{\pi}{2}}^{y_{lo}} \frac{a(y)}{(g(y)+\mu^2)^{\frac{\alpha}{2}}} - \frac{a(y)}{(g(y))^{\frac{\alpha}{2}}} dy = O(\mu^2). \tag{3.38}$$

Combining these using (3.27) gives the result. □

Remark 3.9. *The main adaptation of this proof relies on the singular behaviour of the integrand - for the case where $\alpha < 1$ we no longer have logarithmic behaviour. This enables us to derive a simpler limit equation in the next section. The initial part of the proof remains the same up to a change in exponent - the main differences will be seen in the estimates that arise from some of the terms.*

We now introduce a result which applies in the proof of Theorem 3.7, where the local minimum is no longer at the origin (the proof follows that of Lemma 3.8).

Corollary 3.10. *Let a and g be smooth periodic functions of period π. Assume that g is non-negative and has a non-degenerate local minimum at x_0 (that is $g'(x_0) = 0$ and $g''(x_0) > 0$), and that $\frac{g(x)}{|sin(x-x_0)|^2} > c$ and $g(x_0) = 0$. Then there exists c' such that for $0 < \mu < c'$ we have*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a(y)}{(g(y)+\mu^2)^{\alpha/2}} dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a(y)}{(g(y))^{\alpha/2}} dy + a(x_0)\left(\frac{1}{2}g''(x_0)\right)^{-1/2} O(\mu^{1-\alpha}) + Error \tag{3.39}$$

where $|Error| \leq c''\mu^2$. The constants c' and c'' depend only on the constants c and bounds on both a and g.

The next lemma follows trivially from the corresponding result in Fefferman and Rodrigo (2012). The proof of this result relies on standard differentiation techniques and the bounds on g when $x_0(\delta) = 0$, and utilises the periodicity assumptions on a and g for the other cases.

Lemma 3.11. *Let $a(y, \delta)$ and $g(y, \delta)$ be smooth functions on $\mathbb{R}/\pi\mathbb{Z} \times \mathbb{R}$. For small δ, assume that $x_0(\delta)$ corresponds to a non-degenerate local minimum of g, that is $g_y(x_0(\delta), \delta) = 0$ and $g_{yy}(x_0(\delta), \delta) > 0$. Assume also that $g(x_0(\delta), \delta) = 0$, $\frac{g(y,\delta)}{|\sin(y-x_0(\delta)|^2} > c$ and that $x_0(\delta)$ depends smoothly on δ. Then*

$$I(\delta) = \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a(y,\delta)}{(g(y,\delta))^{\alpha/2}} \mathrm{d}y \tag{3.40}$$

is a smooth function of δ. *Moreover, we have* $I(\delta) = I(0) + I'(0)\delta + O(\delta^2)$ *for* $|\delta| < c'$ *with*

$$I'(0) = \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_\delta(y,0) + x_0'(0)a_y(y,0)}{(g(y,0))^{\alpha/2}} - \frac{\alpha}{2}\frac{a(y,0)[g_\delta(y,0) + x_0'(0)g_y(y,0)]}{(g(y,0))^{(\alpha+2)/2}} \mathrm{d}y. \tag{3.41}$$

3.4 The Limit Equation

Combining equations (3.11) and (3.15) - (3.17), and noting that (3.18) - (3.20) are error terms as outlined on page 35, we currently have the following form of the limit equation:

$$\begin{aligned}
o(1) = \Omega_\tau &+ \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}\left[-(1+\varphi'^2(s))R_\tau^{-1} - \varphi'\varphi_\tau\right]\Omega_s \\
&- \frac{1}{\delta}\frac{\varphi_\tau}{(1+\varphi'(s))^{1/2}}\Omega_\xi + 2\frac{\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2}\xi\Omega_\xi \qquad (3.42)\\
&- \frac{L^2}{Det(s)}\iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_\xi\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \qquad (3.43)\\
&+ \frac{L\delta}{Det(s)}\iint K_\alpha^\delta \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\left[\Omega_\xi\Omega_{\bar{s}} - \Omega_s\Omega_{\bar{\xi}}\right]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \qquad (3.44)\\
&+ \frac{L^2\delta}{Det(s)}\iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\left(\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\right.\\
&\left. + \frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}\right)\Omega_\xi\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \qquad (3.45)
\end{aligned}$$

where the notation $o(1)$ is used to denote terms that are zero in the limit as $\delta \to 0$, and the kernel is given by:

$$K_\alpha^\delta(s,\bar{s},\xi,\bar{\xi}) = \frac{\chi(K_R)}{\left(K_R^2 + K_\varphi^2\right)^{\alpha/2}} \tag{3.46}$$

where for simplicity we have defined the following:

$$K_R = R^{-1}(s) - R^{-1}(\bar{s}) - \frac{\varphi'(s)\xi\delta}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'(\bar{s})\bar{\xi}\delta}{(1+\varphi'^2(\bar{s}))^{1/2}},$$

$$K_\varphi = \varphi(s) - \varphi(\bar{s}) + \frac{\xi\delta}{(1+\varphi'^2(s))^{1/2}} - \frac{\bar{\xi}\delta}{(1+\varphi'^2(\bar{s}))^{1/2}}$$

and:

$$K_\alpha^0 = \frac{\chi(R^{-1}(s) - R^{-1}(\bar{s}))}{[(R^{-1}(s) - R^{-1}(\bar{s}))^2 + (\varphi(s) - \varphi(\bar{s}))^2]^{\alpha/2}}. \tag{3.47}$$

We now aim to utilise the results from the previous section to study terms (3.43)-(3.45) that appear in $u \cdot \nabla q$ as previously described.

Proof of Theorem 3.7 Following the techniques applied in Fefferman and Rodrigo (2012), on fixing s, ξ and $\bar{\xi}$ we define the following function $G(\bar{s}, \delta)$ that appears in the denominator of the kernel K_α^δ, that is:

$$G(\bar{s}, \delta) = K_R^2(\bar{s}, \delta) + K_\varphi^2(\bar{s}, \delta), \tag{3.48}$$

and for δ fixed, let G attain its minimum at some point $s_0(\delta)$ where s_0 is a smooth function of δ. Setting:

$$\mu^2 = G(s_0(\delta), \delta), \quad g(\bar{s}, \delta) = G(\bar{s}, \delta) - G(s_0(\delta), \delta)$$

we see that the terms arising from $u \cdot \nabla q$ all contain integrals of the form:

$$\int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a(\bar{s}, \delta)}{(G(\bar{s}, \delta))^{\alpha/2}} \mathrm{d}\bar{s} = \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a(\bar{s}, \delta)}{(g(\bar{s}, \delta) + \mu^2)^{\alpha/2}} \mathrm{d}\bar{s},$$

for which the combined lemmas give:

$$\begin{aligned} \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a(\bar{s}, \delta)}{(g(\bar{s}, \delta) + \mu^2)^{\alpha/2}} \mathrm{d}\bar{s} =& a(s_0(\delta), \delta) \left(\frac{1}{2} g_{\bar{s}\bar{s}}(s_0(\delta), \delta) \right)^{-1/2} O(\mu^{1-\alpha}) + I(0) \\ &+ I'(0)\delta + O(\mu^2) + O(\delta^2). \end{aligned} \tag{3.49}$$

The function $a(\bar{s}, \delta)$ for each of the terms will automatically be periodic in $\bar{s}$. We first show that g satisfies the assumptions for use of Lemma 3.8 and 3.11 and derive the terms required in determining each of the terms in (3.49). We have:

$$
\begin{aligned}
&g_{\bar s}(\bar s,\delta) = G_{\bar s}(\bar s,\delta)\\
&= 2\left(R^{-1}(\bar s - R^{-1}(s) - \frac{\varphi'(s)\xi\delta}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'(\bar s)\bar\xi\delta}{(1+\varphi'^2(\bar s))^{1/2}}\right)\\
&\times\left(-R_{\bar s}^{-1}(\bar s) + \frac{\delta\bar\xi\varphi''(\bar s)R_{\bar s}^{-1}(\bar s)}{(1+\varphi'^2(\bar s))^{1/2}} - \frac{\delta\bar\xi\varphi'^2(\bar s)\varphi''(\bar s)R_{\bar s}^{-1}(\bar s)}{(1+\varphi'^2(\bar s))^{3/2}}\right)\\
&+2\left(\varphi(s)-\varphi(\bar s) + \frac{\xi\delta}{(1+\varphi'^2(s))^{1/2}} - \frac{\bar\xi\delta}{(1+\varphi'^2(\bar s))^{1/2}}\right)\\
&\times\left(-\varphi'(\bar s)R_{\bar s}^{-1}(\bar s) + \frac{\delta\bar\xi\varphi'(\bar s)\varphi''(\bar s)R_{\bar s}^{-1}(\bar s)}{(1+\varphi'^2(\bar s))^{3/2}}\right).
\end{aligned}
$$

We note that we have a local minumum of g when $s_0(\delta) = s$, that is $g_{\bar s}(s,\delta) = G_{\bar s}(s,\delta) = 0$; the function g has been defined to satisfy automatically the condition $g(s,\delta) = 0$. We have:

$$
\begin{aligned}
\mu^2 &= \left(-\frac{\varphi'(s)\xi\delta}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'(\bar s)\bar\xi\delta}{(1+\varphi'^2(\bar s))^{1/2}}\right)^2 + \left(\frac{\xi\delta}{(1+\varphi'^2(s))^{1/2}} - \frac{\bar\xi\delta}{(1+\varphi'^2(\bar s))^{1/2}}\right)^2\\
&= \frac{1}{(1+\varphi'^2(s))}[\delta^2\varphi'^2(s)(\xi-\bar\xi)^2 + \delta^2(\xi-\bar\xi)^2]\\
&= \delta^2(\xi-\bar\xi)^2.
\end{aligned}
$$

Remark 3.12. *As previously noted, the structure of the almost sharp front in the smooth case requires the integrals in $\bar\xi$ to be taken over the range $[-1,1]$. Hence, for fixed ξ when taking the limit as δ approaches zero, we have that μ^2 is of order δ^2 where this term appears. In the analytic case we see that the second kernel needs to be used in deriving the limit equation; in this case $\bar\xi \in \mathbb{R}$, and we use the exponential decay of the kernel to control the growth of any terms of the form $\delta\bar\xi$ that appear.*

We also have the following properties of g:

$$
\begin{aligned}
&G_\delta(\bar s,\delta)\\
&= 2\left(R^{-1}(\bar s - R^{-1}(s) - \frac{\varphi'(s)\xi\delta}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'(\bar s)\bar\xi\delta}{(1+\varphi'^2(\bar s))^{1/2}}\right)\\
&\times\left(-\frac{\varphi'(s)\xi}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'(\bar s)\bar\xi}{(1+\varphi'^2(\bar s))^{1/2}}\right)
\end{aligned}
$$

$$+2\Big(\varphi(s) - \varphi(\bar{s}) + \frac{\xi\delta}{(1+\varphi'^2(s))^{1/2}} - \frac{\bar{\xi}\delta}{(1+\varphi'^2(\bar{s}))^{1/2}}\Big)$$
$$\times\Big(\frac{\xi}{(1+\varphi'^2(s))^{1/2}} - \frac{\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}}\Big),$$

$$G_\delta(s,\delta) = 2\delta^2(\bar{\xi} - \xi)^2,$$

and so:

$$\begin{aligned} g_\delta(\bar{s},0) = 2\Big(R^{-1}(s) - R^{-1}(\bar{s})\Big)\Big(-\frac{\varphi'(s)\xi}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}}\Big) \\ + 2\Big(\varphi(s) - \varphi(\bar{s})\Big)\Big(\frac{\xi}{(1+\varphi'^2(s))^{1/2}} - \frac{\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}}\Big). \end{aligned} \tag{3.50}$$

A series of elementary calculations gives the second derivatives of g. In particular the term $g_{\bar{s}\bar{s}}(s,\delta)$ is required for our estimates. In Fefferman and Rodrigo (2012), using a geometric series expansion it is shown that:

$$\Big(\frac{1}{2}g_{\bar{s}\bar{s}}(s_0(\delta),\delta)\Big)^{-1/2} = \frac{1}{L}\Big(1 + \frac{1}{2}\delta(\bar{\xi}) + \xi)\frac{\varphi''(s)}{(1+\varphi'^2(s))^{3/2}} + O(\delta^2)\Big) \tag{3.51}$$

and that $s_0(\delta) = s$ is precisely a non-degenerate minimum.

We now use these terms and the previous results to give the estimates for terms (3.43)-(3.45); for this purpose we introduce the following functions:

$$a_1(\bar{s},\delta) = L^2\frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\chi(K_R), \tag{3.52}$$

$$a_2(\bar{s},\delta) = L\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}[\Omega_\xi\Omega_{\bar{s}} - \Omega_s\Omega_{\bar{\xi}}]\chi(K_R), \tag{3.53}$$

$$\begin{aligned} a_3(\bar{s},\delta) = L^2\frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\chi(K_R) \\ \times\Big(\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} + \frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}\Big)\Omega_{\bar{\xi}}, \end{aligned} \tag{3.54}$$

for which we note that $a_1(s,\delta) = a_3(s,\delta) = 0$.

(3.43)

This term can be written as:

$$
\begin{aligned}
&-\frac{1}{Det(s)}\Bigg(\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times[-1,1]} \frac{a_1(\bar{s},\delta)}{(g(\bar{s},\delta)+\mu^2)^{\alpha/2}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Bigg)\Omega_\xi \\
&= -\frac{1}{Det(s)}\int\limits_{[-1,1]}\Bigg(\int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_1(\bar{s},\delta)}{(g(\bar{s},\delta)+\mu^2)^{\alpha/2}}\mathrm{d}\bar{s}\Bigg)\mathrm{d}\bar{\xi}\Omega_\xi
\end{aligned}
\tag{3.55}
$$

where on applying Lemmas 3.8 and 3.11, with $s_0'(0)=0$, we have:

$$
\begin{aligned}
&\int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_1(\bar{s},\delta)}{(g(\bar{s},\delta)+\mu^2)^{\alpha/2}}\mathrm{d}\bar{s} \\
&= \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_1(\bar{s},0)}{(g(\bar{s},0))^{\alpha/2}}\mathrm{d}\bar{s} + \delta\int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_{1\delta}(\bar{s},0)}{(g(\bar{s},0))^{\alpha/2}}\mathrm{d}\bar{s} - \delta\int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{\alpha}{2}\frac{a_1(\bar{s},0)g_\delta(\bar{s},0)}{(g(\bar{s},0))^{(\alpha+2)/2}}\mathrm{d}\bar{s} \\
&\qquad + O(\mu^2) + O(\delta^2) \\
&= \int\limits_{\mathbb{R}/\pi\mathbb{Z}} L^2\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s} \\
&+\delta\int\limits_{\mathbb{R}/\pi\mathbb{Z}} L^2\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\chi'(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}} \\
&\qquad\times\Bigg(-\frac{\varphi'(s)\xi}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}}\Bigg)\Omega_{\bar{\xi}}\mathrm{d}\bar{s} \\
&-\delta\alpha\int\limits_{\mathbb{R}/\pi\mathbb{Z}} L^2\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Bigg(-\frac{\varphi'(s)\xi}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}}\Bigg) \\
&\qquad\times\frac{(R^{-1}(s)-R^{-1}(\bar{s}))\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{(\alpha+2)/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s} \\
&-\delta\alpha\int\limits_{\mathbb{R}/\pi\mathbb{Z}} L^2\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Bigg(\frac{\xi}{(1+\varphi'^2(s))^{1/2}}-\frac{\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}}\Bigg) \\
&\qquad\times\frac{(\varphi(s)-\varphi(\bar{s}))\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{(\alpha+2)/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s} \\
&\qquad+o(1),
\end{aligned}
$$

and for ease of notation we introduce the function $H(s,\bar{s},\xi,\bar{\xi})$ so that the final two terms can be written as

$$-\delta\alpha \int\limits_{\mathbb{R}/\pi\mathbb{Z}} L^2 H(s,\bar{s},\xi,\bar{\xi})\Omega_{\bar{\xi}}\mathrm{d}\bar{s},$$

noting that for $\alpha < 1$, the function H is integrable.

(3.44)

The coefficient $\frac{L\delta}{Det(s)}$ has been shown to be O(1) in δ. In this case $a_2(s,\delta)$ is non-zero; we show that the term in which this appears contributes to the error terms. We have:

$$\begin{aligned}
&\int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_2(\bar{s},\delta)}{(g(\bar{s},\delta)+\mu^2)^{\alpha/2}}\mathrm{d}\bar{s} \\
&= \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_2(\bar{s},0)}{(g(\bar{s},0))^{\alpha/2}}\mathrm{d}\bar{s} + a_2(s,\delta)\frac{1}{L}\left(1+\frac{1}{2}\delta(\bar{\xi}+\xi)\frac{\varphi''(s)}{(1+\varphi'^2(s))^{3/2}} + O(\delta^2)\right)O(\mu^{1-\alpha}) \\
&\quad + O(\mu^2) \\
&= \int\limits_{\mathbb{R}/\pi\mathbb{Z}} L\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}[\Omega_\xi\Omega_{\bar{s}} - \Omega_s\Omega_{\bar{\xi}}] \\
&\quad \times \frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\mathrm{d}\bar{s} \\
&\quad + o(1)
\end{aligned}$$

where, on integrating over the domain $\bar{\xi} \in [-1,1]$ with ξ fixed, as previously remarked, the second term is $O(\delta^{2-2\alpha})$; the coefficient $a(s,\delta)$ is $O(1)$.

(3.45)

Similarly for this term; the coefficient is O(1) and so we only need an expansion to this order, that is:

$$\begin{aligned}
&\int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_3(\bar{s},\delta)}{(g(\bar{s},\delta)+\mu^2)^{\alpha/2}}\mathrm{d}\bar{s} \\
&= \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{a_3(\bar{s},0)}{(g(\bar{s},0))^{\alpha/2}}\mathrm{d}\bar{s} + O(\mu^2) + O(\delta^2)
\end{aligned}$$

$$= \int\limits_{\mathbb{R}/\pi\mathbb{Z}} L^2 \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \frac{\chi(R^{-1}(s) - R^{-1}(\bar{s}))}{[(R^{-1}(s) - R^{-1}(\bar{s}))^2 + (\varphi(s) - \varphi(\bar{s}))^2]^{\alpha/2}}$$
$$\times \left(\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} + \frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}} \right) \Omega_{\bar{\xi}} \mathrm{d}\bar{s}$$
$$+ o(1).$$

Combining these we have:

$$\begin{aligned}
o(1) = \Omega_\tau &+ \frac{1}{L} \frac{1}{(1+\varphi'^2(s))^{1/2}} \left[-(1+\varphi'^2(s)) R_\tau^{-1} - \varphi'\varphi_\tau \right] \Omega_s \\
&- \frac{1}{\delta} \frac{\varphi_\tau}{(1+\varphi'(s))^{1/2}} \Omega_\xi + 2 \frac{\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2} \xi \Omega_\xi \qquad (3.56) \\
&- \frac{1}{Det(s)} \iint L^2 \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \\
&\times \frac{\chi(R^{-1}(s) - R^{-1}(\bar{s}))}{[(R^{-1}(s) - R^{-1}(\bar{s}))^2 + (\varphi(s) - \varphi(\bar{s}))^2]^{\alpha/2}} \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \Omega_\xi \qquad (3.57) \\
&- \frac{\delta}{Det(s)} \iint L^2 \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \left(- \frac{\varphi'(s)\xi}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}} \right) \\
&\times \frac{\chi'(R^{-1}(s) - R^{-1}(\bar{s}))}{[(R^{-1}(s) - R^{-1}(\bar{s}))^2 + (\varphi(s) - \varphi(\bar{s}))^2]^{\alpha/2}} \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \Omega_\xi \qquad (3.58) \\
&- \frac{\delta\alpha}{Det(s)} \iint L^2 H(s, \bar{s}, \xi, \bar{\xi}) \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \Omega_\xi \qquad (3.59) \\
&+ \frac{\delta}{Det(s)} \iint L \frac{1 + \varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} [\Omega_\xi \Omega_{\bar{s}} - \Omega_s \Omega_{\bar{\xi}}] \\
&\times \frac{\chi(R^{-1}(s) - R^{-1}(\bar{s}))}{[(R^{-1}(s) - R^{-1}(\bar{s}))^2 + (\varphi(s) - \varphi(\bar{s}))^2]^{\alpha/2}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \qquad (3.60) \\
&+ \frac{\delta}{Det(s)} \iint L^2 \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \left(\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} + \frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}} \right) \\
&\times \frac{\chi(R^{-1}(s) - R^{-1}(\bar{s}))}{[(R^{-1}(s) - R^{-1}(\bar{s}))^2 + (\varphi(s) - \varphi(\bar{s}))^2]^{\alpha/2}} \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \Omega_\xi \qquad (3.61)
\end{aligned}$$

where all double integrals are taken over the cylindrical domain $\mathbb{R}/\mathbb{Z} \times [-1, 1]$.

Recalling the approximations for $\frac{1}{Det(s)}$ (3.10), we have:

$$o(1) = \Omega_\tau + \frac{1}{L} \frac{1}{(1+\varphi'^2(s))^{1/2}} \left[-(1+\varphi'^2(s)) R_\tau^{-1} - \varphi'\varphi_\tau \right] \Omega_s \qquad (3.62)$$
$$- \frac{1}{\delta} \frac{\varphi_\tau}{(1+\varphi'^2(s))^{1/2}} \Omega_\xi + \qquad (3.63)$$

$$2\frac{\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2}\xi\Omega_\xi \tag{3.64}$$

$$-\frac{1}{L\delta}\iint L^2\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}$$
$$\times\frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi \tag{3.65}$$

$$+\frac{1}{L}\frac{\varphi''(s)}{(1+\varphi'^2(s))}\xi\iint L^2\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}$$
$$\times\frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi \tag{3.66}$$

$$-\frac{1}{L}\iint L^2\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\left(-\frac{\varphi'(s)\xi}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}}\right)$$
$$\times\frac{\chi'(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi \tag{3.67}$$

$$-\frac{\alpha}{L}\iint L^2 H(s,\bar{s},\xi,\bar{\xi})\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi \tag{3.68}$$

$$+\frac{1}{L}\iint L\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}[\Omega_\xi\Omega_{\bar{s}}-\Omega_s\Omega_{\bar{\xi}}]$$
$$\times\frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{3.69}$$

$$+\frac{1}{L}\iint L^2\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\left(\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}+\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}\right)$$
$$\times\frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi. \tag{3.70}$$

We note that the terms (3.63) and (3.65) cancel due to the sharp front equation (3.21) and using $\int_1^1 \Omega_{\bar{\xi}}\mathrm{d}\bar{\xi} = 1$. Using properties of the integral of Ω as stated in Remark 3.1, we are able to take the formal limit as $\delta \to 0$ and obtain the limit equation:

$$0 = \Omega_\tau + \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}\left[-(1+\varphi'^2(s))R_\tau^{-1} - \varphi'\varphi_\tau\right]\Omega_s$$
$$+2\frac{\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2}\xi\Omega_\xi \tag{3.71}$$

$$+\frac{\varphi''(s)}{(1+\varphi'^2(s))}\xi\int L\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}$$
$$\times\frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\mathrm{d}\bar{s}\Omega_\xi \tag{3.72}$$

$$
\begin{aligned}
&-\iint L\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\left(-\frac{\varphi'(s)\xi}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{1/2}}\right)\\
&\qquad\times\frac{\chi'(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi} &(3.73)\\
&\qquad-\alpha\iint LH(s,\bar{s},\xi,\bar{\xi})\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi} &(3.74)\\
&\qquad+\iint\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}[\Omega_{\xi}\Omega_{\bar{s}}-\Omega_s\Omega_{\bar{\xi}}]\\
&\qquad\times\frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} &(3.75)\\
&+\iint L\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\left(\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}+\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}\right)\\
&\qquad\times\frac{\chi(R^{-1}(s)-R^{-1}(\bar{s}))}{[(R^{-1}(s)-R^{-1}(\bar{s}))^2+(\varphi(s)-\varphi(\bar{s}))^2]^{\alpha/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi} &(3.76)
\end{aligned}
$$

of the form in Theorem 3.7. ☐

We note that the final form of the limit equation for $\alpha < 1$ is non-linear in Ω, however this form is much simpler than that for the limit equations when $\alpha = 1$ (see §3.2).

For the SQG equations, well-posedness of the corresponding limit equation remains an open problem; one suggestion for this could be to study the well-posedness of (3.25) first. If we could achieve existence of smooth approximate solutions to this equation then we could attempt to study a limiting procedure as $\alpha \to 1$, using standard energy methods. This approach has been studied extensively for solutions to the Euler and Navier-Stokes equations as $\nu \to 0$, see for example Majda and Bertozzi (2002) and Kato (1984). This would allow us to utilise the results proved in Chapter 2.

In Fefferman and Rodrigo (2012), the limit equation for $\alpha = 1$, stated in (3.23), contains a term of the form:

$$\frac{2}{L}\Omega_{\xi}(s,\xi)\int\Omega_{\bar{s}}(s,\bar{\xi})\log(|\bar{\xi}-\xi|)\mathrm{d}\bar{\xi} \tag{3.77}$$

which is Prandtl-like. For the case of $0 < \alpha < 1$, the limit equation contains a term (included in (3.75)) of the form:

$$\Omega_{\xi}\iint K(s,\bar{s})\Omega_{\bar{s}}(\bar{s},\bar{\xi})\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{3.78}$$

whose structure is also Prandtl-like, noting that in the Prandtl case the kernel

K would be 1 and so much simpler. Unfortunately, the behaviour of such terms currently prevents us from proving existence to the limiting equation in the smooth case.

We can however show that for the corresponding limit equation for $\alpha < 1$ when considering analytic functions, we have an existence and uniqueness result for a related IVP. In the next chapter we present such a result for approximate solutions in short time using a version of the abstract Cauchy-Kovaleskaya (ACK) theorem.

Chapter 4

Limit Equations in the Analytic Case

Following on from the results of Chapter 3, we now focus on the construction of almost-sharp fronts for analytic solutions. We continue to study the α-equation on the two-dimensional cylindrical domain, defining a family of functions, asymptotic to almost-sharp fronts, parametrised by $\delta > 0$ as in (3.3). For this family, for each fixed α, we give a derivation of a limit equation as δ approaches 0 and prove an approximation result analogous to Theorem 3.7.

In order to derive the limit equation, we introduce an adapted change of coordinates from those given in the previous chapter; this will be outlined in the next section. We show that, in order to construct the solutions in the analytic case, it is necessary for φ to satisfy the sharp-front equation in the form (2.11). In §4.2 we give some preliminary lemmas required to complete the estimates and we complete the derivation of the limit equation in §4.3.

In §4.4 we prove an existence and uniqueness result for approximate solutions to the α-equation. We outline a version of the Abstract Cauchy-Kovalevskaya (ACK) Theorem, see Safonov (1995), and show that on application of this theorem for a suitable class of functions, there exist solutions (of the form constructed) in short time[1].

[1] We have chosen the most applicable version of the ACK theorem for the system derived within this chapter; for other statements of this theorem see for example Friedman (1961), Caflisch (1990) or Nirenberg (1972).

4.1 Change of Coordinates

In order to derive the limit equations we first define an appropriate change of coordinates for the analytic case; the mapping below was introduced in Fefferman and Rodrigo (2011a). Here we give an overview of the change of variables required; the calculations are provided in detail in Appendix C.

Using the renormalised arc length $R(x,t)$ given in (3.4) we construct a mapping from the fixed domain $(s,\xi) \in [0,1) \times \mathbb{R}$:

$$(x,y) = (R^{-1}(s,t), \varphi(R^{-1}(s,t),t)) + \left(n_1(R^{-1}(s,t)) \frac{\delta\xi}{1+100(\delta\xi)^{100}}, n_2(R^{-1}(s,t))\delta\xi \right) \tag{4.1}$$

where n_1 and n_2 are the first and second components of the normal to the curve φ, that is:

$$n_1(R^{-1}(s,t)) = \frac{-\varphi'((R^{-1}(s,t)),t)}{(1+\varphi'^2(s))^{1/2}} \quad \text{and} \quad n_2 = \frac{1}{(1+\varphi'^2(s))^{1/2}}. \tag{4.2}$$

Remark 4.1. *Recall that in the smooth case we defined a mapping between the fixed domain $(s,\xi) \in [0,1) \times [-1,1]$ and a δ-neighbourhood of the curve φ (3.5), and that for $\delta \leq \delta_0$ the map was injective. In the analytic case presented here, where ξ takes values in $\mathbb{R}$ (see below), we now require an injective mapping from $(s,\xi) \in [0,1) \times \mathbb{R}$. The map presented in (4.1) is guaranteed to be injective for small δ; there exists $\tilde{\delta_0}$ depending on the curvature of φ such that (4.1) is injective for $\delta \leq \tilde{\delta_0}$.*

We introduce a new time variable τ and study a family of solutions to the α-equation, $q(x,y,t) = \Omega(s,\xi,\tau)$, parametrised by δ. Such solutions will be aysmptotic, by the above mapping, to the almost sharp fronts introduced in Chapter 3. In addition we need to impose some further assumptions on the function Ω; for this purpose we define the following function spaces:

Definition 4.2. *For $a,b \in \mathbb{Z}^+$ such that $|a+b| \geq 1$ the function space $\Pi_{a,b}$ is given by:*

$$\begin{aligned} \Pi_{a,b} = \{\Omega \in C^\omega : \exists C > 0,\ |\partial_s^b \partial_\xi^a \Omega| \leq C(1+|\xi|)^{-100-a-b} \quad &\text{for} \quad |\xi| > 1\} \\ \cap \{|\Omega - {}^1\!/\!_2| \leq C(1+|\xi|)^{-100} \quad \text{for} \quad \xi > 1\}& \\ \cap \{|\Omega + {}^1\!/\!_2| \leq C(1+|\xi|)^{-100} \quad \text{for} \quad \xi < -1\}.& \end{aligned} \tag{4.3}$$

Remark 4.3. *We can also ensure that by defining the function Ω symmetrically* *that we achieve the property* $\int_{\mathbb{R}} \Omega(\bar{s},\bar{\xi})\mathrm{d}\bar{\xi} = -\int_{\mathbb{R}} \bar{\xi}\Omega_{\bar{\xi}}(\bar{s},\bar{\xi})\mathrm{d}\bar{\xi}$ *and* $\int_{\mathbb{R}} \Omega_{\bar{\xi}}(\bar{s},\bar{\xi})\mathrm{d}\bar{\xi} = 1$.

Recall that for the smooth case we were able to restrict to the case $\xi \in [-1,1]$ due to the structure of the almost sharp fronts. For the analytic case presented here, assuming Ω is analytic in all variables, there exists no such interval on which we can assume Ω is constant outside of the interval; we then require $\xi \in \mathbb{R}$. We impose the conditions of Definition 4.2 on Ω; that is we have some nice decay properties and outside of $\xi \in [-1,1]$ the function is close to the constants given in the definition of the almost-sharp front. By restricting the function Ω to function spaces of this type we are able to simplify the estimates on integral terms that arise in the derivation of the limit equation. In particular, for such terms we are able to prove a smallness condition, ensuring that we need only consider the integration over the domain $|\bar{\xi}| \leq \delta^{-\theta}$ for some $\theta \in (0,1)$. This condition will be introduced in §4.2.

Remark 4.4. *We could instead consider function spaces with different rates of decay, for example:*

$$\begin{aligned}\Pi_{a,b,c,l} = \{&\Omega \in C^{\omega} : \exists C > 0,\ |\partial_s^b \partial_\xi^a \Omega| \leq C(1+c|\xi|)^{-l-a-b} \quad \text{for} \quad |\xi| > 1\} \\ &\cap \{|\Omega - 1/2| \leq C(1+c|\xi|)^{l} \quad \text{for} \quad \xi > 1\} \\ &\cap \{|\Omega + 1/2| \leq C(1+c|\xi|)^{l} \quad \text{for} \quad \xi < -1\}\end{aligned}$$

where the ranges of l and c can be studied in more detail. The decay rate chosen in Definition 4.2 is enough to give a smallness condition as needed in the derivation of the limit equations.

Returning to the change of variables, we now write the α-equation in terms of $\Omega(s,\xi,\tau)$, where:

$$\begin{aligned}(x,y,t) &= (R^{-1}(s,\tau), \varphi(R^{-1}(s,\tau),\tau),\tau) \\ &+ \left(n_1(R^{-1}(s,\tau))\frac{\delta\xi}{1+100(\delta\xi)^{100}}, n_2(R^{-1}(s,\tau))\delta\xi\right).\end{aligned} \tag{4.4}$$

We first have, as in the smooth case, that:

$$\partial_x = \frac{1}{Det(s)}\frac{\partial y}{\partial \xi}\partial_s - \frac{1}{Det(s)}\frac{\partial y}{\partial s}\partial_\xi, \quad \partial_y = \frac{1}{Det(s)}\frac{\partial x}{\partial s}\partial_\xi - \frac{1}{Det(s)}\frac{\partial x}{\partial \xi}\partial_s,$$
$$\partial_t = \frac{I}{Det(s)}\partial_s + \frac{II}{Det(s)}\partial_\xi + \partial_\tau,$$

where:

$$Det(s) = \frac{\partial x}{\partial s}\frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial \xi}, \quad I = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \tau} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \tau}, \quad II = \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial \tau}\frac{\partial x}{\partial s},$$

with:

$$\frac{\partial x}{\partial s} = R_s^{-1} + \left[\frac{-\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\right]\frac{\delta\xi}{1+100(\delta\xi)^{100}}, \tag{4.5}$$

$$\frac{\partial y}{\partial s} = \varphi' R_s^{-1} - \frac{\varphi'\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\delta\xi, \tag{4.6}$$

$$\frac{\partial x}{\partial \xi} = -\frac{\varphi'}{(1+\varphi'^2(s))^{1/2}}\left[\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right], \tag{4.7}$$

$$\frac{\partial y}{\partial \xi} = R_\tau^{-1} + \frac{\delta\xi}{1+100(\delta\xi)^{100}}\left[\frac{-\varphi'' R_\tau^{-1} - \varphi'_\tau}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2(\varphi'' R_\tau^{-1} + \varphi'_\tau)}{(1+\varphi'^2(s))^{3/2}}\right], \tag{4.8}$$

$$\frac{\partial x}{\partial \tau} = R_\tau^{-1} + \frac{\delta\xi}{1+100(\delta\xi)^{100}}\left[\frac{-\varphi'' R_\tau^{-1} - \varphi'_\tau}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2(\varphi'' R_\tau^{-1} + \varphi'_\tau)}{(1+\varphi'^2(s))^{3/2}}\right], \tag{4.9}$$

$$\frac{\partial y}{\partial \tau} = \varphi' R_\tau^{-1} + \varphi_\tau - \frac{\varphi'(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^{3/2}}. \tag{4.10}$$

Using $R^{-1} = \frac{L}{(1+\varphi'^2(s))^{1/2}}$ we have the determinant:

$$\begin{aligned} Det(s) =& L\delta\Bigg[1 - \frac{\varphi''\delta\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})} - \frac{\varphi'^2 100(\delta\xi)^{100}}{(1+\varphi'^2(s))(1+100(\delta\xi)^{100})} \\ &- \frac{\varphi'^2 10^4(\delta\xi)^{100}}{(1+\varphi'^2(s))(1+100(\delta\xi)^{100})^2} + \frac{\varphi'^2\varphi'' 10^4(\delta\xi)^{101}}{(1+\varphi'^2(s))^{5/2}(1+100(\delta\xi)^{100})^2}\Bigg] \end{aligned} \tag{4.11}$$

and the remaining terms:

$$\begin{aligned} I =& \frac{\delta}{(1+\varphi'^2(s))^{1/2}}[-\varphi'\varphi_\tau - (1+\varphi'^2)R_\tau^{-1}] + \frac{100(\delta\xi)^{100}(\varphi'^2 R_\tau^{-1} + \varphi'\varphi_\tau)\delta}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})} \\ &+ \frac{(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta^2\xi}{(1+\varphi'(s)^2)(1+100(\delta\xi)^{100})} + \frac{\varphi' 10^4(\delta\xi)^{100}\delta(\varphi' R_\tau^{-1} + \varphi_\tau)}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})^2} \\ &- \frac{\varphi'^2 10^4(\delta\xi)^{101}\delta(\varphi'' R_\tau^{-1} + \varphi'_\tau)}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})^2}, \end{aligned} \tag{4.12}$$

$$II=-\varphi_\tau\frac{L}{(1+\varphi'^2(s))^{1/2}}+\frac{\delta\xi L\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}$$
$$+\frac{100(\delta\xi)^{101}L\varphi'\varphi'_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}. \tag{4.13}$$

In order to rewrite the α-equation in the new coordinates we need to give the forms for the time and spatial derivatives. For this we use that, for fixed ξ, the following estimates apply:

$$\frac{1}{Det(s)}=\frac{1}{L\delta}+\frac{1}{L}\frac{\varphi''\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})}+o(1).$$

When calculating $\partial_t q$ we need the forms of $\frac{I}{Det(s)}$ and $\frac{II}{Det(s)}$; we use the above estimate to calculate the former:

$$\frac{I}{Det(s)}=\frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}[-\varphi'\varphi_\tau-(1+\varphi'^2)R_\tau^{-1}]+o(1). \tag{4.14}$$

For the latter, that is the coefficient of the Ω_ξ term, we do not provide an estimate at present; we see instead that when estimating integrals that arise from $u\cdot\nabla q$ that the sharp front equation provides some cancellation. We therefore have:

$$\begin{aligned}\partial_t q&=\frac{I}{Det(s)}\Omega_s+\frac{II}{Det(s)}\Omega_\xi+\Omega_\tau\\&=\frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}[-\varphi'\varphi_\tau-(1+\varphi'^2)R_\tau^{-1}]\Omega_s\\&+\frac{1}{Det(s)}\Bigg(-\varphi_\tau\frac{L}{(1+\varphi'^2(s))^{1/2}}+\frac{\delta\xi L\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\\&+\frac{100(\delta\xi)^{101}L\varphi'\varphi'_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\Bigg)\Omega_\xi\\&+\Omega_\tau+o(1).\end{aligned} \tag{4.15}$$

We turn our attention to the term $u\cdot\nabla q$, where u is as defined in (1.12) and, for the analytic case, can be written as a convolution with the kernel $\tilde{K}_\alpha$ as given in (2.7). Let $\tilde{K}_\alpha^\delta$ be this kernel under the new coordinates, highlighting the dependence on δ; that is using the same notation as in §3.4 we have:

$$\tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})=\frac{1}{(\cosh(K_\varphi)-\cos(K_R))^{\alpha/2}} \tag{4.16}$$

with:

$$\tilde{K}^0_\alpha(s,\bar{s},\xi,\bar{\xi}) = \frac{1}{(\cosh(\varphi(s)-\varphi(\bar{s}))-\cos(R^{-1}(s)-R^{-1}(\bar{s})))^{\alpha/2}} \tag{4.17}$$

and so:

$$u(s,\xi,\tau) = \iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})\nabla^\perp_{\bar{s},\bar{\xi}}\Omega(\bar{s},\bar{\xi})\cdot\nabla_{s,\xi}\Omega(s,\xi)Det(\bar{s})\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}. \tag{4.18}$$

The expansion for $u\cdot\nabla q$ is given in Appendix C, (C.26) - (C.54). Recalling that $\tilde{K}^\delta_\alpha \in L^1$ whose denominator behaves exponentially, for fixed $\xi\in\mathbb{R}$, the majority of these integral terms contribute to the error o(1). The exceptions to this are precisely the terms (C.28), (C.30) and (C.32) which will require further analysis to be carried out in §4.2. Combining these terms with the expression for $\partial_t q$ in (4.15), we obtain the following form of the limit equation:

$$\begin{aligned}
o(1) = {}& \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}[-\varphi'\varphi_\tau-(1+\varphi'^2)R^{-1}_\tau]\Omega_s && (4.19)\\
&+\frac{1}{Det(s)}\left(-\varphi_\tau\frac{L}{(1+\varphi'^2(s))^{1/2}}+\frac{\delta\xi L\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\right.\\
&\left.+\frac{100(\delta\xi)^{101}L\varphi'\varphi'_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\right)\Omega_\xi+\Omega_\tau && (4.20)\\
&-\frac{L^2}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_\xi\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} && (4.21)\\
&+\frac{\delta L}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_\xi-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} && (4.22)\\
&+\frac{\delta L^2}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\\
&\times\left[\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}+\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\right]\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}, && (4.23)
\end{aligned}$$

where the double integrals are taken over the domain $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$. For fixed $\xi\in\mathbb{R}$ we have:

- The third term in (4.20), that is the third coefficient of Ω_ξ that arises from the time derivative, is an error term.
- The integral in (4.21) has a coefficient that is $O(\frac{1}{\delta})$ which blows up as $\delta\to 0$.

We show that no such behaviour occurs in the limit equation; on rearranging this term appropriately we have some cancellation with the first term in (4.20) as a result of the sharp-front equation (2.11).

- The coefficients of terms (4.22) and (4.23) are $O(1)$. On estimating these terms we show that we obtain the terms on setting $\delta = 0$ and some $o(1)$ terms. For the first of these, we obtain a cancellation with the second term in (4.20) and so no terms of the form $\xi\Omega_\xi$ appear in the limit equation.

Finally we introduce the sharp front equation for the kernel $\tilde{K}_\alpha$ under the new coordinates:

$$\frac{\partial\varphi}{\partial\tau}(s,\tau) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi'(s) - \varphi'(\bar{s})}{[\cosh(\varphi(s) - \varphi(\bar{s})) - \cos(R^{-1}(s) - R^{-1}(\bar{s}))]^{\alpha/2}} \frac{L}{(1 + \varphi'^2(\bar{s}))^{1/2}} \mathrm{d}\bar{s}. \tag{4.24}$$

Having achieved the current form of the limit equation (4.19)-(4.23), the next two sections contain estimates on the terms highlighted above; in completing these we are able to take a formal limit and prove the following:

Theorem 4.5. *Given a curve $y = \varphi(x,t)$ satisfying the sharp front equation (2.11), and a family of functions $\Omega(s,\xi,\tau)$ (indexed by $\delta > 0$) defined by the curve φ via the change of coordinates given in (4.1), such that for each τ we have:*

- *Ω is an analytic function of s and ξ belonging to the function spaces $\Pi_{0,1}$ and $\Pi_{1,0}$ with the* sup-*norm bounded independent of δ,*
- *$\Omega(s,\xi,0) = \Omega_0$.*

Then Ω is an approximate solution of the α-equation if and only if it solves the equation:

$$\Omega_\tau + A_1\Omega_s + \int\limits_{\mathbb{R}/\mathbb{Z}} Q_1 \int\limits_{\mathbb{R}} \Omega(\bar{s},\bar{\xi})\mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_\xi + \int\limits_{\mathbb{R}/\mathbb{Z}} Q_2 \int\limits_{\mathbb{R}} \Omega_{\bar{s}}(\bar{s},\bar{\xi})\mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_\xi = 0 \tag{4.25}$$

where A_1, Q_1 and Q_2 can be explicitly computed and are independent of ξ and Ω; $Q_1, Q_2 \in L^1(\mathbb{R}/\mathbb{Z})$.

4.2 Derivation of the Limit Equation

We now give the proof of Theorem 4.5; we first group together and rewrite some of the terms that appear in (4.19)-(4.23), showing that the estimates on these terms can be reduced to showing just three additional results - a smallness condition restricting the domain of integration, and two estimates analogous to that of Lemma 3.8. We have the following:

2nd term (4.20) + 1st term (4.23)

We notice that the addition of these two terms results in some cancellation and in fact will leave a term of $o(1)$ plus an error term; this removes the second term of (4.20) from the limit equation and hence removes the term of the form $\xi\Omega_\xi$, simplifying the existence result that follows in §4.4. Rewriting the term from (4.20) we have:

$$
\begin{aligned}
&\frac{\delta\xi L\varphi''(s)\varphi_\tau(s)}{Det(s)}\frac{1}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\Omega_\xi \\
&= \frac{\delta\xi L\varphi''(s)\varphi_\tau(s)\Omega_\xi}{Det(s)(1+\varphi'^2(s))^2} + \frac{\delta\xi L\varphi''(s)\varphi_\tau(s)}{Det(s)(1+\varphi'^2(s))^2}\left[\frac{1}{(1+100(\delta\xi)^{100})}-1\right]\Omega_\xi
\end{aligned}
$$

where the second term is clearly an error. The sharp front equation (4.24) then gives:

$$
\begin{aligned}
&\frac{\delta\xi L\varphi''(s)\varphi_\tau(s)}{Det(s)}\frac{1}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100}}\Omega_\xi \\
&= \frac{\delta\xi L\varphi''(s)}{Det(s)(1+\varphi'^2(s))^2}\int\limits_{\mathbb{R}/\mathbb{Z}}\tilde{K}^0_\alpha\frac{L(\varphi'(s)-\varphi'(\bar{s}))}{(1+\varphi'^2(\bar{s}))^{1/2}}\mathrm{d}\bar{s}\Omega_\xi + o(1) \\
&= \frac{\delta\xi L^2\varphi''(s)}{Det(s)(1+\varphi'^2(s))^2}\int\limits_{\mathbb{R}/\mathbb{Z}}\tilde{K}^0_\alpha\frac{\varphi'(s)-\varphi'(\bar{s})}{(1+\varphi'^2(\bar{s}))^{1/2}}\mathrm{d}\bar{s}\Omega_\xi + o(1) \\
&= -\frac{\delta\xi L^2\varphi''(s)}{Det(s)(1+\varphi'^2(s))^2}\int\limits_{\mathbb{R}/\mathbb{Z}}\tilde{K}^0_\alpha\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(\bar{s}))^{1/2}}\mathrm{d}\bar{s}\Omega_\xi + o(1) \\
&= -\frac{\delta\xi L^2\varphi''(s)}{Det(s)(1+\varphi'^2(s))^2}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}^0_\alpha\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi + o(1),
\end{aligned}
$$

using that $\tilde{K}^0_\alpha$ is independent of $\bar{\xi}$, and by construction $\int\limits_{\mathbb{R}}\Omega_{\bar{\xi}}\mathrm{d}\bar{\xi} = 1$. The sum of the two terms can then be simplified, and so it remains to estimate the following:

$$\frac{\delta\xi L^2\varphi''(s)}{Det(s)(1+\varphi'^2(s))^2}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}(\tilde{K}_\alpha^\delta-\tilde{K}_\alpha^0)\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi. \tag{4.26}$$

2nd term (4.23)

We first rewrite this term as a difference, that is:

$$\frac{\delta L^2}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha^\delta\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi =$$

$$\frac{\delta L^2}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}(\tilde{K}_\alpha^\delta-\tilde{K}_\alpha^0)\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi \tag{4.27}$$

$$+\frac{\delta L^2}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha^0\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi. \tag{4.28}$$

We will show that the first term contributes to the error term. The second term in $O(1)$ and so will remain in the limit equation. Noting by Remark 4.3 that $\int\Omega(s,\xi)\mathrm{d}\xi=-\int\xi\Omega_\xi(s,\xi)\mathrm{d}\xi$, and integrating over ξ we simplify the expression so that (4.28) is as follows:

$$\begin{aligned}
&-\frac{\delta L^2}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha^0\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\varphi''(\bar{s})\Omega}{(1+\varphi'^2(\bar{s}))^{3/2}}\mathrm{d}\bar{s}\Omega_\xi\\
&=-L\left(\frac{\delta L}{Det(s)}-1\right)\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha^0\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\varphi''(\bar{s})\Omega}{(1+\varphi'^2(\bar{s}))^{3/2}}\mathrm{d}\bar{s}\Omega_\xi\\
&\quad-L\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha^0\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\varphi''(\bar{s})\Omega}{(1+\varphi'^2(\bar{s}))^{3/2}}\mathrm{d}\bar{s}\Omega_\xi\\
&=-L\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha^0\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\varphi''(\bar{s})\Omega}{(1+\varphi'^2(\bar{s}))^{3/2}}\mathrm{d}\bar{s}\Omega_\xi \qquad (4.29)\\
&\qquad\qquad+o(1),
\end{aligned}$$

using the properties of the determinant outlined in §4.1.

(4.22)

We again write this term as a difference:

$$\frac{\delta L}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}^{\delta}_{\alpha}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_{\xi}-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}=$$

$$\frac{\delta L}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}(\tilde{K}^{\delta}_{\alpha}-\tilde{K}^{0}_{\alpha})\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_{\xi}-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{4.30}$$

$$+\frac{\delta L}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}^{0}_{\alpha}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_{\xi}-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}$$

where we are able to show that (4.30) gives an error term. The second term is equal to:

$$\left(\frac{\delta L}{Det(s)}-1\right)\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}^{0}_{\alpha}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_{\xi}-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}$$
$$+\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}^{0}_{\alpha}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_{\xi}-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi},$$

of which the first term is automatically an error. By integrating over $\bar{\xi}$ we write the second term of size $O(1)$ in a more optimal form, that is:

$$\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}^{0}_{\alpha}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_{\xi}-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}=$$

$$\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}}\tilde{K}^{0}_{\alpha}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi} \tag{4.31}$$

$$-\int\limits_{\mathbb{R}/\mathbb{Z}}\tilde{K}^{0}_{\alpha}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\mathrm{d}\bar{s}\Omega_s. \tag{4.32}$$

1st term (4.20) + (4.21)

We pair these together and use the sharp front equation to show that this is an error term. In particular this shows that no terms with blow-up behaviour appear and so we are able to take a formal limit. We have:

$$-\frac{1}{Det(s)}\varphi_{\tau}(s)\frac{L}{(1+\varphi'^2(s))^{1/2}}\Omega_{\xi}$$

$$-\frac{L^2}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \tilde{K}^\delta_\alpha \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi$$
$$=-\frac{L^2}{Det(s)}\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} (\tilde{K}^\delta_\alpha-\tilde{K}^0_\alpha)\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi. \tag{4.33}$$

In order to derive the limit equation in the form of Theorem 4.5, it remains to show that (4.26), (4.29), (4.30) and (4.33) are indeed error terms. The coefficient of both (4.26) and (4.29) are $O(1)$; the integrand of these is of a similar form to (4.33). The coefficient of (4.33) is $O(\frac{1}{\delta})$ and so to show that (4.26), (4.29) and (4.33) are error terms it suffices to show that:

$$\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} (\tilde{K}^\delta_\alpha-\tilde{K}^0_\alpha)\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi = o(\delta). \tag{4.34}$$

This automatically proves that (4.33) is an error term, but given that both Ω and φ are bounded, the other estimates follow in the same manner. The coefficient of (4.30) is $O(1)$ and so we also need to show that:

$$\iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} (\tilde{K}^\delta_\alpha-\tilde{K}^0_\alpha)\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_\xi-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} = o(1). \tag{4.35}$$

In order to prove such estimates we first introduce a smallness condition:

Lemma 4.6. *Given a real analytic function $H(\bar{s},\bar{\xi})$ uniformly bounded in δ, $\alpha,\beta\in\mathbb{N}$, $\theta\in(0,1)$ and $\Omega\in\Pi_{\alpha,\beta}$, then:*

$$\int\limits_{|\bar{\xi}|\geq\frac{1}{\delta^\theta}} H(\bar{s},\bar{\xi})\partial^\alpha_{\bar{s}}\partial^\beta_{\bar{\xi}}\Omega\mathrm{d}\bar{\xi} = o(1). \tag{4.36}$$

Proof To prove the above estimate we show the following:

$$\int\limits_{\frac{1}{\delta^\theta}}^{\infty} H(\bar{s},\bar{\xi})\partial^\alpha_{\bar{s}}\partial^\beta_{\bar{\xi}}\Omega\mathrm{d}\bar{\xi} = o(1). \tag{4.37}$$

The remaining calculation is analogous. We have, using the assumptions and a suitable change of coordinates:

$$\left|\int\limits_{\frac{1}{\delta^\theta}}^{\infty} H(\bar{s},\bar{\xi})\partial_{\bar{s}}^{\alpha}\partial_{\bar{\xi}}^{\beta}\Omega \mathrm{d}\bar{\xi}\right| \le \int\limits_{\frac{1}{\delta^\theta}}^{\infty}\left|H(\bar{s},\bar{\xi})\partial_{\bar{s}}^{\alpha}\partial_{\bar{\xi}}^{\beta}\Omega\right|\mathrm{d}\bar{\xi}$$

$$\le \|H\|_\infty \int\limits_{\frac{1}{\delta^\theta}}^{\infty} |\partial_{\bar{\xi}}^{\alpha}\partial_{\bar{\xi}}^{\beta}\Omega|\mathrm{d}\bar{\xi} \le \|H\|_\infty \int\limits_{\frac{1}{\delta^\theta}}^{\infty} \frac{1}{(1+|\xi|)^{100+\alpha+\beta}}\mathrm{d}\bar{\xi}$$

$$\le \|H\|_\infty \int\limits_{\frac{1}{\delta^\theta}}^{\infty} \frac{1}{|\xi|^{100+\alpha+\beta}}\mathrm{d}\bar{\xi} = \|H\|_\infty \int\limits_{\delta^{1-\theta}} \frac{\delta^{100+\alpha+\beta}}{x^{100+\alpha+\beta}}\frac{\mathrm{d}x}{\delta}$$

$$\le C\delta^{\theta(99+\alpha+\beta)},$$

which converges to 0 when $0 < \theta < 1$. □

Setting $H = \tilde{K}_\alpha^\delta - \tilde{K}_\alpha^0$, which by definition is analytic and uniformly bounded in δ on $|\bar{\xi}| \ge \frac{1}{\delta^\theta}$, an application of Lemma 4.6 to the integrals in (4.34) and (4.35) gives the required estimates on the outer domain. It suffices to prove the remainder of the estimates on the domain $|\bar{\xi}| < \frac{1}{\delta^\theta}$. The remaining estimates will give us a further restriction on $\theta(\alpha)$.

The following lemmas are the analogues of Lemma 3.8 for the analytic case; the result presented in §3.2 does not directly apply here as it required some smoothness on the numerator of the integrand. When Ω is assumed to be analytic we can no longer differentiate the numerator in the final terms that need estimating; we now prove these directly.

Lemma 4.7. *Let φ be as defined in Chapter 2 and let Ω satisfy the assumptions of Theorem 4.5. Given $0 < \theta < \frac{1-\alpha}{3} < 1$, over the domain of integration $\mathbb{R}/\mathbb{Z} \times \{|\bar{\xi}| < \frac{1}{\delta^\theta}\}$ we have:*

$$\iint (\tilde{K}_\alpha^\delta - \tilde{K}_\alpha^0)\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi = o(\delta). \tag{4.38}$$

Proof We split the integral in $\bar{s}$ into an inner and outer region given by $|s-\bar{s}| < \delta$ and $|s-\bar{s}| \ge \delta$ respectively. With the kernels as defined and noting that the integrand evaluated as s is 0, the Lebegue differentiation theorem gives:

$$\lim_{\delta\to 0}\frac{1}{\delta}\iint\limits_{|s-\bar{s}|<\delta} (\tilde{K}_\alpha^\delta - \tilde{K}_\alpha^0)\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi = 0, \tag{4.39}$$

giving the required estimate on the inner region.

To study the integral on the outer region we introduce a new kernel κ_α^δ with the same properties of $\tilde{K}_\alpha^\delta$, and present the proof of the above estimate for this kernel. The result remains true for the original kernel using precisely the same method displayed below; we present the calculations for the new kernel for simplicity. Setting:

$$\kappa_\alpha^\delta(s,\bar{s},\xi,\bar{\xi}) = \frac{1}{[(s-\bar{s})^2+\delta^2(\xi-\bar{\xi})^2]^{\alpha/2}} \tag{4.40}$$

it suffices to show that:

$$\int_{|s-\bar{s}|\geq\delta} (\kappa_\alpha^\delta - \kappa_\alpha^0)\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\mathrm{d}\bar{s} = O(\delta^b) \tag{4.41}$$

for some $b>1$. Note that we are on the domain $|\bar{\xi}| < \frac{1}{\delta^\theta}$ and so $|\delta^\theta\bar{\xi}| < 1$, where we require that this term is bounded.

Given $\varphi \in C^\infty$, the mean value theorem gives the following bound on $\varphi'(\bar{s}) - \varphi'(s)$:

$$|\varphi'(\bar{s})-\varphi'(s)| \leq \sup_{\tilde{s}}|\varphi''(\tilde{s})||\bar{s}-s|.$$

By the assumptions on both φ and Ω, taking bounds on the integrand in (4.41), we need only consider the following:

$$\begin{aligned}
&\iint_{|s-\bar{s}|\geq\delta}(\kappa_\alpha^\delta-\kappa_\alpha^0)|\bar{s}-s|\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\\
&\leq \int_{|s-\bar{s}|\geq\delta}\left|\frac{s-\bar{s}}{[(s-\bar{s})^2+\delta^2(\xi-\bar{\xi})^2]^{\alpha/2}} - \frac{s-\bar{s}}{[(s-\bar{s})^2]^{\alpha/2}}\right|\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\\
&= \iint_{|s-\bar{s}|\geq\delta}\left|\frac{s-\bar{s}}{[(s-\bar{s})^2+\delta^{2(1-\theta)}(\delta^\theta\xi-\delta^\theta\bar{\xi})^2]^{\alpha/2}} - \frac{s-\bar{s}}{[(s-\bar{s})^2]^{\alpha/2}}\right|\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}.
\end{aligned} \tag{4.42}$$

Set $G(\eta) = \frac{s-\bar{s}}{[(s-\bar{s})^2+\eta]^{\alpha/2}}$ so that the integrand in (4.42) can be written as $|G(\eta)-G(0)|$, with $\eta = \delta^{2(1-\theta)}(\delta^\theta\xi-\delta^\theta\bar{\xi})^2$, and:

$$G'(\eta) = -\frac{\alpha}{2}\frac{s-\bar{s}}{[(s-\bar{s})^2+\eta]^{(\alpha+2)/2}}.$$

By a second application of the mean value theorem we have:

$$|G(\eta) - G(0)| \leq \sup_{\tilde{\eta}\in(0,\eta)} |G'(\tilde{\eta})|\tilde{\eta} \leq \frac{\alpha}{2}\frac{|(s-\bar{s})|}{|s-\bar{s}|^{\alpha+2}}\eta \tag{4.43}$$

which, for η as above, ξ fixed and $\delta^\theta\bar{\xi}$ bounded we have:

$$\begin{aligned}(4.38) &\leq C \iint\limits_{|s-\bar{s}|\geq\delta} \frac{\delta^{2(1-\theta)}}{|s-\bar{s}|^{\alpha+1}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \\ &\leq C\delta^{-\theta}\delta^{2(1-\theta)}\delta^{-\alpha} = C\delta^{2-3\theta-\alpha} \end{aligned}\tag{4.44}$$

for some constant C independent of δ and for which the exponent is greater than 1 for the precise choice of θ given. This proves the estimate. $\square$

Lemma 4.8. *Let φ be as defined in Chapter 2 and let Ω satisfy the assumptions of Theorem 4.5. Given $0 \leq \theta < \frac{1-\alpha}{3} < 1$, over the domain of integration $\mathbb{R}/\mathbb{Z} \times \{|\bar{\xi}| < \frac{1}{\delta^\theta}\}$ we have:*

$$\iint (\tilde{K}^\delta_\alpha - \tilde{K}^0_\alpha)\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}(\Omega_{\bar{s}}\Omega_\xi - \Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} = o(1). \tag{4.45}$$

Proof Splitting the integral the same way as in Lemma 4.7, the estimate here holds for the outer region $|s-\bar{s}| \geq \delta$ using the same method. The only difference being that we lose precisely 1 power in the exponent of δ in the final step due to the numerator of the integrand in (4.45). This gives the $o(1)$ estimate for the same choice of θ.

For the inner region we have that $\tilde{K}^\delta_\alpha \in L^1$ and the difference $\tilde{K}^\delta_\alpha - \tilde{K}^0_\alpha$ dominated by an L^1 function. In particular we have there exists an upper bound M such that:

$$|K^\delta_\alpha| \leq \frac{M}{|s-\bar{s}|^\alpha} \tag{4.46}$$

and so there exists $\tilde{M}$ such that:

$$|\tilde{K}^\delta_\alpha - \tilde{K}^0_\alpha| \leq \frac{\tilde{M}}{|s-\bar{s}|^\alpha} \in L^1 \quad \text{for} \quad \alpha < 1. \tag{4.47}$$

Using also that $\delta\bar{\xi}$ is bounded and ξ fixed, an application of the Lebesgue dominated convergence theorem gives that:

$$\lim_{\delta\to 0} \iint\limits_{|s-\bar{s}|<\delta} (\tilde{K}^{\delta}_{\alpha} - \tilde{K}^{0}_{\alpha}) \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} (\Omega_{\bar{s}}\Omega_{\xi} - \Omega_{\bar{\xi}}\Omega_{s}) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \to 0,$$

which completes the result. $\square$

Taking the formal limit in (4.19) - (4.23) as $\delta \to 0$, using Lemmas 4.6-4.8, we obtain the limit equation:

$$\begin{aligned} \Omega_{\tau} + \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}} \left[-\varphi'\varphi_{\tau} - (1+\varphi'^2)R_{\tau}^{-1}\right] \Omega_{s} \\ -L \iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \tilde{K}^{0}_{\alpha} \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \frac{\varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}} \Omega(\bar{s},\bar{\xi}) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi} \\ + \iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \tilde{K}^{0}_{\alpha} \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi} \\ - \int\limits_{\mathbb{R}/\mathbb{Z}} \tilde{K}^{0}_{\alpha} \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \mathrm{d}\bar{s}\Omega_{s} \ = 0 \end{aligned} \tag{4.48}$$

of the form required in Theorem 4.5, with:

$$\begin{aligned} A_1 &= \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}} \left[-\varphi'\varphi_{\tau} - (1+\varphi'^2)R_{\tau}^{-1}\right] \\ &\quad - \int\limits_{\mathbb{R}/\mathbb{Z}} \tilde{K}^{0}_{\alpha} \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \mathrm{d}\bar{s}, \\ Q_1 &= -L\tilde{K}^{0}_{\alpha} \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \frac{\varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}}, \\ Q_2 &= \tilde{K}^{0}_{\alpha} \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \end{aligned}$$

noting that Q_1 is an analytic function and $Q_2 \in L^1$. $\square$

4.3 Abstract Cauchy-Kovalevskaya Theorem

We first give an overview of the ACK theorem before applying this to the limit equation derived in (4.48). We use the form of the theorem due to Safonov (1995), and the notation introduced in Sammartino and Caflisch (1998a).

Definition 4.9. *A Banach Scale* $\{X_\rho : 0 < \rho < \rho_0\}$ *with norms* $\|\cdot\|_\rho$ *is a collection of Banach spaces such that* $X_{\rho'} \subset X_{\rho''}$ *and* $\|\cdot\|_{\rho''} \leq \|\cdot\|_{\rho'}$ *whenever* $\rho'' \leq \rho' \leq \rho_0$.

In the following chapters we consider ρ to be a vector of parameters - (ρ, r) (Chapter 4), (ρ, m) (Chapter 5) and (ρ, θ) (Chapter 6).

Remark 4.10. *As will be highlighted in the construction of the function spaces and Banach scales in §4.4.1, §5.2 and §6.2, the parameters correspond to the size of the domains of analyticity in the variables* s *and* ξ *respectively. We see that we only consider functions analytic in* s *on a strip of width* ρ*, where the second parameter corresponds to the domain in which the functions are analytic in* ξ *and is defined differently for each of the remaining existence proofs. In this respect there is no direct connection between the two parameters. The proof of the ACK theorem remains the same when considering a vector of parameters (Sammartino and Caflisch, 1998a).*

Definition 4.11. *Let* $\tau > 0$, $0 < \rho \leq \rho_0$ *and* $R > 0$. *Then we define the following:*

1. $X_{\rho,\tau}$ *is the set of all functions* $u(t)$ *from* $[0, \tau]$ *to* X_ρ *endowed with the norm*

$$\|u\|_{\rho,\tau} = \sup_{0 \leq t \leq \tau} \|u(t)\|_\rho.$$

2. $Y_{\rho,\beta,\tau}$ *is the set of all functions* $u(t)$ *from* $[0, \tau]$ *to* X_ρ *with the norm*

$$\|u\|_{\rho,\beta,\tau} = \sup_{0 \leq t \leq \tau} \|u(t)\|_{\rho - \beta t}.$$

3. *The notation* $X_{\rho,\tau}(R)$ *and* $Y_{\rho,\beta,\tau}(R)$ *is used to describe the balls of radius* R *in* $X_{\rho,\tau}$ *and* $Y_{\rho,\beta,\tau}$ *respectively, that is*

$$\|u\|_{\rho,\tau} \leq R \quad \text{and} \quad \|u\|_{\rho,\beta,\tau} \leq R.$$

For $t \in [0, T]$, in this and the subsequent chapters, we will be considering systems of the form

$$u_t + F(t, u) = 0, \quad u(0) = 0. \tag{4.49}$$

The existence of solutions to such systems results from the ACK theorem; we use the version due to Safonov (1995).

Theorem 4.12. *Suppose that there exist $R > 0$, $T > 0$, $\rho_0 > 0$ and $\beta > 0$ such that if $0 < t \leq T$ the following holds:*

1. *For every $0 < \rho' < \rho < \rho_0 - \beta_0 T$ and every $u \in X_{\rho,T}(R)$ the function $F(t,u) : [0,T] \to X_{\rho'}$ is continuous.*
2. *For every $0 < \rho \leq \rho_0 - \beta_0 T$ the function $F(t,0) : [0,T] \to X_{\rho,T}(R)$ is continuous in $[0,T]$ and*
$$||F(t,0)||_{\rho_0-\beta_0 T} \leq R_0 < R. \tag{4.50}$$
3. *For every $0 < \rho' < \rho(s) \leq \rho_0 - \beta_0 s$ and every $u_1, u_2 \in Y_{\rho_0,\beta_0,T}(T)$ we have*
$$||F(t,u) - F(t,v)||_{\rho'} \leq \frac{C}{\rho - \rho'}||u - v||_{\rho}. \tag{4.51}$$

Then there exist $\beta > \beta_0$ and $T^* > 0$ such that

$$u_t + F(t,u) = 0 \tag{4.52}$$

has a unique solution in Y_{ρ_0,β,T^*}.

Remark 4.13. *To complete the existence result that follows in the remainder of the thesis, the ACK theorem presented above is of the optimal form. This is preferable to using the standard Cauchy-Kovalevskaya (CK) theorem which ensures, under the assumptions of analyticity, existence of a unique analytic solution in a neighbourhood of the origin to a Cauchy problem of the form (4.49). The analyticity assumption in Theorem 4.12 is contained in the Cauchy estimates defined in (4.51). These estimates are the natural analogues of the Cauchy estimates that an analytic function satisfies, which is why they do not appear in the CK theorem. A summary of the CK theorem and the ACK theorem are contained in Appendix A (§A.2).*

4.4 Existence of Approximate Solutions

In this section we introduce the main result of this chapter, the existence and uniqueness of approximate solutions to the α-equation. These solutions are asymptotic to almost-sharp fronts and take the form constructed in §4.1. The existence result will follow from a direct application of the ACK theorem (Theorem 4.12).

We first define the function spaces on which we prove the result; these spaces will be $L^2(s, \xi)$ and are the natural function spaces for the α-equation (Sammartino and Caflisch, 1998a). We then define an IVP for the equation derived in (4.48), rewriting the equation in a Banach Space to be introduced in §4.4.2. For such a system we verify the assumptions of the ACK theorem, including the required Cauchy estimates; this forms §4.4.3.

4.4.1 Function Spaces

By complexifying the space variables only, that is $s \in \mathbb{C}$, $\xi \in \mathbb{C}$ and $\tau \in \mathbb{R}$, we study functions that are analytic and L^2 in s and ξ. Define the strip as follows:

$$D(\rho) = \mathbb{R} \times (-\rho, \rho) = \{x \in \mathbb{C} : \Im x \in (-\rho, \rho)\} \tag{4.53}$$

and the corresponding path along which the L^2 integration is performed:

$$\Gamma(b) = \{x \in \mathbb{C} : \Im x = b\}. \tag{4.54}$$

In the function spaces that follow, the parameter l that counts the number of derivatives in both s and ξ will be restricted to $l \geq 4$ as in Sammartino and Caflisch (1998a).

Definition 4.14. *For $f(s)$ analytic on the strip $D(\rho)$, for some $\rho > 0$, the norm $\|f\|_\rho$ is given by:*

$$\|f\|_\rho = \sup_{|y| \leq \rho} \|f(\cdot + iy)\|_{L^2(\mathbb{R}/\mathbb{Z})}. \tag{4.55}$$

Definition 4.15. *Given $l \in \mathbb{N}$ and $\rho > 0$, $H^{l,\rho}$ is the set of all complex functions $f(s)$ such that:*

- *f is analytic in $D(\rho)$ and periodic in $\Re s$.*
- *$\partial_s^\alpha f \in L^2(\Gamma(\Im s))$ for $|\Im s| < \rho$, $\alpha \leq l$; i.e. if $|\Im s| < \rho$, then $\partial_s^\alpha f(\Re s + i\Im s)$ is a square integrable function of $\Re s$.*
- *The norm $\|f\|_{l,\rho}$ is finite, where:*

$$\|f\|_{l,\rho} = \sum_{\alpha=0}^{l} \|\partial_s^\alpha f\|_\rho. \tag{4.56}$$

Definition 4.16. *Given $l \in \mathbb{N}$, $\rho > 0$ and $r > 0$, $H^{l,\rho,r}$ is the set of all complex functions $f(s, \xi)$ such that:*

- *f is analytic in $D(\rho) \times D(r)$ and periodic in $\Re s$.*
- $\partial_\xi^{\alpha_1}\partial_s^{\alpha_2} f \in L^2(\Gamma(r'); H^{0,\rho})$ *with* $|r'| \le r$, $\alpha_1 + \alpha_2 \le l$.
- *The norm $\|f\|_{l,\rho,r}$ is finite, where:*

$$\|f\|_{l,\rho,r} = \sum_{\alpha_1=0}^{l} \sum_{\alpha_2=0}^{l-\alpha_1} \sup_{|r'|\le r} \left\| \|\partial_s^{\alpha_1}\partial_\xi^{\alpha_2} f(\cdot,\xi)\|_{0,\rho} \right\|_{L^2(\Gamma(r'))}. \tag{4.57}$$

For a fixed time t the functions included in the existence proof will belong to the above spaces. We now define the required spaces for functions that also depend on time.

Definition 4.17. *Given $l \in \mathbb{N}$, $\rho, r, \beta, T > 0$ the function $f(s,\xi,t)$ is in $H^{l,\rho,r}_{\beta,T}$ if and only if*

- *$f(s,\xi,t)$ is periodic in $\Re s$ and $\Re\xi$ and analytic in $D(\rho) \times D(r)$.*
- $\partial_t^k f \in C([0,T]; H^{l-k,\rho,r})$ *for* $0 \le k \le l$.
- *The norm $\|f\|_{l,\rho,r,\beta,T}$ is finite, where:*

$$\|f\|_{l,\rho,r,\beta,T} = \sum_{k=0}^{l} \sup_{0\le t\le T} \|\partial_t^k f(\cdot,\cdot,t)\|_{l-k,\rho-\beta t,r-\beta t}. \tag{4.58}$$

4.4.2 Existence Theorem

With the notation as previously defined we wish to show existence of solutions to the following initial value problem:

$$\begin{cases} \Omega_t + A_1\Omega_s + \int\limits_{\mathbb{R}/\mathbb{Z}} Q_1 \int\limits_{\mathbb{R}} \Omega(\bar{s},\bar{\xi})\mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_\xi + \int\limits_{\mathbb{R}/\mathbb{Z}} Q_2 \int\limits_{\mathbb{R}} \Omega_{\bar{s}}(\bar{s},\bar{\xi})\mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_\xi = 0 \\ \Omega|_{t=0} = \Omega_0 \end{cases} \tag{4.59}$$

where we now use t (replacing τ) to denote the time variable to simplify any further notation.

Using the ACK theorem as outlined in §4.3 and the function spaces as in §4.4.1, in this section will prove the following result:

Theorem 4.18 (Existence of Approximate Solutions)**.** *Let $\Omega_0 \in H^{l,\rho,r}$ and $l \ge 4$. Then the limit equation (4.59) has a unique solution $\Omega \in H^{l,\rho_0,r_0}_{\beta_0,T}$ for some $0 < \rho_0 < \rho, 0 < r_0 < r, \beta_0 > 0$ and $T > 0$.*

The construction of Ω as described in §4.1 requires us to study the function spaces $\Pi_{a,b}$ (Definition 4.2). When applying Theorem 4.12 we study Banach spaces - at present $\Pi_{a,b}$ is not closed under addition. We recast the IVP in (4.59) so that we consider functions that do belong to a Banach space and prove existence for an equivalent system. Define instead:

Definition 4.19. *For $a, b \in \mathbb{N}$ such that $|a+b| \geq 1$ the function space $\tilde{\Pi}_{a,b}$ is given by:*

$$\begin{aligned}\tilde{\Pi}_{a,b} = \{W \in C^{\omega} : \exists C > 0,\ &|\partial_s^b \partial_\xi^a W| \leq C(1+|\xi|)^{-100-a-b} \quad \text{for} \quad |\xi| > 1\} \\ &\cap \{|W| \leq C(1+|\xi|)^{-100} \quad \text{for} \quad \xi > 1\} \\ &\cap \{|W| \leq C(1+|\xi|)^{-100} \quad \text{for} \quad \xi < -1\}\end{aligned} \tag{4.60}$$

which is now closed under addition. If we select $\Omega_0 \in \Pi_{a,b}$ and $W \in \tilde{\Pi}_{a,b}$ and write $\Omega = \Omega_0 + W$, so that the initial condition in (4.59) is satisfied by taking $W|_{t=0} = 0$, then Theorem 4.18 follows if we can show existence to the equivalent system:

$$\begin{cases} W_t + F(t, W) = 0 \\ \quad W|_{t=0} = 0 \end{cases} \tag{4.61}$$

where:

$$\begin{aligned}F(t, W) = A_1 W_s + A_1 \Omega_{0,s} + B_1 W_\xi + \int Q_1 \int W \mathrm{d}\bar{\xi}\mathrm{d}\bar{s} W_\xi \\ + \int Q_1 \int W \mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_{0,\xi} + \int Q_1 \int \Omega_0 \mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_{0,\xi} + \int Q_2 \int W_{\bar{s}} \mathrm{d}\bar{\xi}\mathrm{d}\bar{s} W_\xi \\ + \int Q_2 \int W_{\bar{s}} \mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_{0,\xi} + \int Q_2 \int \Omega_{0,\bar{s}} \mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_{0,\xi}\end{aligned} \tag{4.62}$$

and:

$$B_1 = \int Q_1 \int \Omega_0 \mathrm{d}\bar{\xi}\mathrm{d}\bar{s} + \int Q_2 \int \Omega_{0,\bar{s}} \mathrm{d}\bar{\xi}\mathrm{d}\bar{s}.$$

It remains to check that F satisfies the assumptions of Theorem 4.12, which we will now verify. The first condition is automatically satisfied by construction, in the norms $H^{l,\rho,r}$. For the second assumptions we need to prove the existence of a constant R_0 such that:

$$\|F(t,0)\|_{l,\rho_0-\beta t,r_0-\beta t} =$$

$$\|A_1\Omega_{0,s} + \int Q_1 \int \Omega_0 \mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_{0,\xi} + \int Q_2 \int \Omega_{0,\bar{s}}\mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_{0,\xi}\|_{l,\rho_0-\beta t,r_0-\beta t} \leq R_0 \quad (4.63)$$

for $0 \leq t \leq T$ and $\Omega_0 \in H^{l,\rho,r}$.

We show that the bounds depend only on norms of Ω_0 and the differences $(\rho-\rho_0)$ and $(r-r_0)$. We return to this estimate, which is trivial, and will follow from several lemmas that we now introduce for proving the Cauchy estimates required in (4.51).

4.4.3 Cauchy Estimates

It remains for us to show that for $W_1, W_2 \in H^{l,\rho_0,r_0}_{\beta,T}$ for $l \geq 4$ and $\rho' < \rho(a) \leq \rho_o - \beta a$, $r' < r(a) \leq r_o - \beta a$ then we have:

$$\|F(t,W_1) - F(t,W_2)\|_{l,\rho',r'} \leq C\bigg(\frac{\|W_1 - W_2\|_{l,\rho(a),r'}}{\rho(a)-\rho'} + \frac{\|W_1 - W_2\|_{l,\rho',r(a)}}{r(a)-r'}\bigg). \quad (4.64)$$

The definition of $F(t,W)$, (4.62), contains derivatives of W of at most first order; we show the following Cauchy Estimates for functions in $H^{l,\rho}$:

Lemma 4.20. *For $f(s)$ analytic in the strip $D(\rho)$ for $\rho > 0$:*

$$\|\partial_s f\|_{\rho'} \leq \frac{\|f\|_\rho}{\rho - \rho'} \quad (4.65)$$

holds for $0 < \rho' < \rho$.

Proof We use the Cauchy Integral Formula to write:

$$f'(s) = f'(x+iy) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-(x+iy)}\mathrm{d}z \quad (4.66)$$

for $x, y \in \mathbb{R}$, where C_r denotes the path $C_r(t) = x + iy + re^{it}, 0 \leq t \leq 2\pi$. We choose $r = \frac{1}{2}(\rho-\rho') > 0$ such that the circle is contained in $|\Im s| < \rho$. Then we have:

$$f'(x+iy) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(x+iy+re^it)re^{it}}{r^2e^{i2t}}\mathrm{d}t$$

and so:

$$
\begin{aligned}
|f'(x+iy)| &\leq \frac{1}{2\pi r}\int\limits_0^{2\pi} |f(x+iy+re^{it})|\mathrm{d}t \\
|f'(x+iy)|^2 &\leq \frac{C}{r^2}\int\limits_0^{2\pi} |f(x+iy+re^{it})|^2\mathrm{d}t \\
\int\limits_{\mathbb{R}/\mathbb{Z}} |f'(x+iy)|^2\mathrm{d}x &\leq \frac{C}{r^2}\int\limits_{\mathbb{R}/\mathbb{Z}}\int\limits_0^{2\pi} |f(x+iy+re^{it})|^2\mathrm{d}t\mathrm{d}x \\
&= \frac{C}{r^2}\int\limits_0^{2\pi}\int\limits_{\mathbb{R}/\mathbb{Z}} |f(x+iy+re^{it})|^2\mathrm{d}x\mathrm{d}t \qquad (4.67)\\
&\leq \frac{C}{r^2}\int\limits_0^{2\pi}\sup_{|b|\leq\rho}|f(x+ib)|^2\mathrm{d}x\mathrm{d}t \leq \frac{C2\pi}{r^2}\|f\|_\rho, \qquad (4.68)
\end{aligned}
$$

where the penultimate step follows by construction of the curve. In particular, for fixed t the inner integral in (4.67) is precisely an integral over a line $\bar{x}+i\bar{y}$ with imaginary part $|\bar{y}| \leq b$ for some $0 \leq b \leq \rho$, as $C_r(t)$ is defined so that it is contained in the strip with imaginary part $\leq \rho$. Hence, taking the supremum over all $|b| \leq \rho$, we obtain the estimate in (4.68). Taking the supremum over $|y| \leq \rho'$ and taking square roots gives:

$$\|f_s\|_{\rho'} \leq \frac{C}{\rho-\rho'}\|f\|_\rho.$$

□

The following lemmas will be utilised for the Cauchy estimates. We first have analogues of several statements in Sammartino and Caflisch (1998a):

Lemma 4.21. *Let $f \in H^{l,\rho}$. For $0 < \rho' < \rho$ then:*

$$\|\partial_s f\|_{l,\rho'} \leq \frac{\|f\|_{l,\rho}}{\rho-\rho'}. \qquad (4.69)$$

which follows immediately from the definition.

Lemma 4.22. *Let $f(s,\xi) \in H^{l,\rho,r}$ with $l \geq 4$ and let $0 < r' < r$, then:*

$$\|f_\xi\|_{l,\rho,r'} \leq \frac{\|f\|_{l,\rho,r}}{r-r'}. \qquad (4.70)$$

and the corresponding:

Lemma 4.23. *Let $f(s,\xi) \in H^{l,\rho,r}$ with $l \geq 4$ and let $0 < \rho' < \rho$, then:*

$$\|f_s\|_{l,\rho',r} \leq \frac{\|f\|_{l,\rho,r}}{\rho - \rho'}. \tag{4.71}$$

which both follow from the definition and proof of Lemma 4.20.

For the function spaces $H^{l,\rho,r}_{\beta,T}$ we have the Sobolev inequality:

Lemma 4.24. *Let $f, g \in H^{l,\rho,r}_{\beta,T}$ and $l \geq 4$. Then $f \cdot g \in H^{l,\rho,r}_{\beta,T}$, and*

$$\|f \cdot g\|_{l,\rho,r,\beta,T} \leq c\|f\|_{l,\rho,r,\beta,T}\|g\|_{l,\rho,r,\beta,T}. \tag{4.72}$$

which combined with the Cauchy estimates above gives:

Lemma 4.25. *Let $f, g \in H^{l,\rho,r}$ with $l \geq 4$, and $0 < \rho' < \rho$. Then:*

$$\|g\partial_s f\|_{l,\rho',r} \leq \|g\|_{l,\rho',r}\frac{\|f\|_{l,\rho,r}}{\rho - \rho'}. \tag{4.73}$$

and

Lemma 4.26. *Let $f, g \in H^{l,\rho,r}$ with $l \geq 4$, and $0 < r' < r$. Then:*

$$\|g\partial_\xi f\|_{l,\rho,r'} \leq \|g\|_{l,\rho,r'}\frac{\|f\|_{l,\rho,r}}{r - r'}. \tag{4.74}$$

The precise form of $F(t, W_1) - F(t, W_2)$ is as follows:

$$\begin{aligned}
&F(t, W_1) - F(t, W_2) = \\
&A_1(W_1 - W_2)_s + B_1(W_1 - W_2)_\xi && \text{(C1)}\\
&+ \iint Q_1(W_1 - W_2)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} W_{1,\xi} + \iint Q_1(W_1 - W_2)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{0,\xi} && \text{(C2)}\\
&+ \iint Q_2(W_1 - W_2)_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} W_{1,\xi} + \iint Q_2(W_1 - W_2)_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{0,\xi} && \text{(C3)}\\
&+ \iint Q_1 W_2\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}(W_1 - W_2)_\xi + \iint Q_2 W_{2,\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}(W_1 - W_2)_\xi, && \text{(C4)}
\end{aligned}$$

for which we require estimate (4.64). Considering each of the terms above we see that those in (C1) satisfy the estimates trivially by lemmas 4.22 and 4.23. The estimate on (C2) follows from the algebra property (4.72) and an application of the following standard result for analytic functions, a version of which is given in Fefferman and Rodrigo (2011b).

Lemma 4.27. *Suppose that Φ in analytic on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ and the function $f(s,\xi) \in H^{l,\rho,r}$ for $l \geq 4$, $\rho, r > 0$. Then:*

$$G(s,\xi) = \iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \Phi(s,\bar{s},\xi,\bar{\xi}) f(\bar{s},\bar{\xi}) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{4.75}$$

satisfies $\|G\|_{l,\rho',r} \leq \frac{\|G\|_{l,\rho,r}}{\rho-\rho'}$ some $\rho' < \rho$.

The terms contained in (C2) are of the form (4.75) with Q_1 analytic on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$; an application of lemma 4.27 gives the required estimates for these two terms. For the estimate on (C3) we again utilise the algebra property of the function spaces under consideration, and lemma 4.27. By construction Q_2 is no longer analytic everywhere and so we need to check the Cauchy estimate of these terms in detail.

Recall that:

$$Q_2 = \tilde{K}_\alpha^0 \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}.$$

We write:

$$\begin{aligned}
&\iint Q_2 W_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \\
&= \iint Q_2 W_{\bar{s}} - K(s-\bar{s}) W_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} + \iint K(s-\bar{s}) W_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi},
\end{aligned} \tag{4.76}$$

where we can choose $K \in L^1$ such that $(Q_2 - K)$ analytic. For example choose $K = \frac{1}{|s-\bar{s}|^\alpha}$. By lemma 4.27 it remains to prove the Cauchy estimates on the second term. We notice that this term can now be written as a convolution and so all derivatives are taken on Ω. We have:

$$\begin{aligned}
&\left\| \iint K(\bar{s}) \Omega_{\bar{s}}(\cdot - \bar{s}) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \right\|_{l,\rho} \\
&= \sum_{\alpha=0}^{l} \left\| \partial_s^\alpha \iint K(\bar{s}) \Omega_{\bar{s}}(\cdot - \bar{s}) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \right\|_{\rho'} = \sum_{\alpha=0}^{l} \left\| \iint K(\bar{s}) \partial_s^\alpha \Omega_{\bar{s}}(\cdot - \bar{s}) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \right\|_{\rho'} \qquad (4.77) \\
&\leq \sum_{\alpha=0}^{l} \iint \|K(\bar{s}) \partial_s^\alpha \Omega_{\bar{s}}(\cdot - \bar{s})\|_{\rho'} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \qquad (4.78) \\
&= \sum_{\alpha=0}^{l} \iint |K(\bar{s})| \|\partial_s^\alpha \Omega_{\bar{s}}(\cdot - \bar{s})\|_{\rho'} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} = \sum_{\alpha=0}^{l} \iint |K(\bar{s})| \|(\partial_s^\alpha \Omega)_s\|_{\rho'} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \qquad (4.79)
\end{aligned}$$

$$\leq \frac{1}{\rho-\rho'}\sum_{\alpha=0}^{l}\iint |K(\bar{s})|\|\partial_s^\alpha\Omega\|_\rho \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{4.80}$$

$$\leq \frac{1}{\rho-\rho'}\sum_{\alpha=0}^{l}\|K\|_{L^1}\|\partial_s^\alpha\Omega\|_\rho = \frac{\|K\|_{L^1}}{\rho-\rho'}\|\Omega\|_{l,\rho}, \tag{4.81}$$

where (4.79) follows using $\partial_s\Omega(s-\bar{s}) = -\partial_{\bar{s}}\Omega(s-\bar{s})$ and (4.81) results from translation invariance.

The estimate on the first term of (C4) follows as for $W_2 \in H^{l,\rho,r}$, $\iint Q_1W_2 \in H^{l,\rho,r}$ using the same trick as in the previous proof. The estimates on the second term in (C4) follow from the standard result that, for two functions A, B analytic in the strip we have, for $l \geq 4$, $0 < \rho' < \rho$, $0 < r' < r$:

$$\|A_sB_\xi\|_{l,\rho',r'} \leq C\left[\|A\|_{l,\rho',r'}\frac{\|B\|_{l,\rho',r}}{r-r'} + \|B\|_{l,\rho',r'}\frac{\|A\|_{l,\rho,r'}}{\rho-\rho'}\right].$$

A combination of all of the estimates given in this section proves (4.59) and so assumption 3 of Theorem 4.12. Returning to the second assumption, using the above lemmas, and the Banach scale property, we also have:

$$\|A_1\Omega_{0,s} + \int Q_1\int \Omega_0\mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_{0,\xi} + \int Q_2\int \Omega_{0,\bar{s}}\mathrm{d}\bar{\xi}\mathrm{d}\bar{s}\Omega_{0,\xi}\|_{l,\rho_0-\beta t,r_0-\beta t} \tag{4.82}$$

$$\leq C\left[\|\Omega_0\|_{l,\rho,r}\frac{\|\Omega_0\|_{l,\rho,r}}{r-r_0} + \|\Omega_0\|_{l,\rho,r}\frac{\|\Omega_0\|_{l,\rho,r}}{\rho-\rho_0}\right] \tag{4.83}$$

as required, where C is a constant depending only on $\|Q_1\|_{L^1}$ and $\|Q_2\|_{L^1}$. This completes the proof of Theorem 4.18. □

We combine the discussion of the results presented here and the existence of exact solutions, to be proved in Chapter 5, and include this in §5.4.

Chapter 5

Existence of Exact Solutions to the α-equation

For the SQG equations, an open question is the existence of a family of almost-sharp front solutions. In this chapter we show that when $\alpha < 1$, under the assumption of analyticity, there exist solutions to the α-equation of the form outlined in Chapter 4. These solutions are asymptotic to almost-sharp fronts.

In Chapter 4 we introduced a change of coordinates in the analytic case in order to derive the limit equations; we now consider the α-equation under this change of coordinates before taking the limit. The precise forms of the terms that are now included in this equation are given in Appendix C, equations (C.13) and (C.26)-(C.54). We study an IVP of the form:

$$\begin{cases} \Omega_\tau + \xi\Omega_\xi + F(\tau, \Omega) = 0 \\ \Omega|_{\tau=0} = \Omega_0 \end{cases} \tag{5.1}$$

Using the version of the ACK theorem as introduced in §4.3, we prove the existence and uniqueness of solutions to (5.1) for short-time. In particular we show that the time of existence is independent of the thickness of the front δ.

Following on from the construction and change of coordinates in the previous chapter, in the next section we derive the precise form of the IVP in (5.1). In §5.2 we define the function spaces on which to apply the ACK theorem (Theorem 4.12). In particular we note that for the term $\xi\Omega_\xi$ that will appear in our IVP, the Cauchy estimates fail when posed on a strip, as previously used, due to the blow-up of ξ. We introduce a new domain - the "bow-tie" - to account for this. In §5.3 we give a statement of the existence result and its derivation. §5.4 contains some discussion of the result.

5.1 Derivation of the Initial Value Problem

In this section we provide a detailed derivation of the IVP for which we prove an existence result; the aim being to show the existence of exact solutions to the α-equation (2.1)-(2.2). In Chapter 4 we considered almost-sharp fronts for this equation under analyticity assumptions. Using a change of coordinates introduced within that chapter with details outlined in Appendix C we were able to rewrite the equation for a scalar function Ω, imposing that Ω belonged to function spaces of the form $\Pi_{a,b}$ (Definition 4.2). For such a family of solutions, dependent on the front thickness δ, we now study the α-equation under the same change of coordinates for fixed δ.

We first consider all of the terms that arise from $\partial_t q$, ∇q and u under the change of variables. We show that we can group many of these together, reducing the lengthy equation that appears in Appendix C to an equation that will just require the study of several different terms.

We assume that δ is small, that is $0 < \delta < \delta_0$. In classifying the type of terms that appear in the IVP the interest is now in the growth of these in δ; this determines the dependence on the interval of the time of existence on δ by standard arguments. We show that the terms arising from $\partial_t q$ and $u \cdot \nabla q$ are all uniformly bounded in δ and hence the maximal time of existence $T > 0$ is indeed independent of δ as claimed in Theorem 5.6 (see §5.3).

The time derivative of q in the new variables (where I and II are defined in (C.10) and (C.11) respectively) is given by:

$$\begin{aligned}
\partial_t q &= \Omega_\tau + \frac{I}{Det(s)}\Omega_s + \frac{II}{Det(s)}\Omega_\xi \\
&= \Omega_\tau + \frac{1}{Det(s)}\Bigg[\frac{\delta}{(1+\varphi'^2(s))^{1/2}}[-\varphi'\varphi_\tau - (1+\varphi'^2)R_\tau^{-1}] \\
&\quad + \frac{100(\delta\xi)^{100}(\varphi'^2 R_\tau^{-1} + \varphi'\varphi_\tau)\delta}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})} + \frac{(\varphi'' R_\tau^{-1} + \varphi_\tau')\delta^2\xi}{(1+\varphi'(s)^2)(1+100(\delta\xi)^{100})} \\
&\quad + \frac{\varphi' 10^4(\delta\xi)^{100}\delta(\varphi' R_\tau^{-1} + \varphi_\tau)}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})^2} - \frac{\varphi'^2 10^4(\delta\xi)^{101}\delta(\varphi'' R_\tau^{-1} + \varphi_\tau')}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})^2}\Bigg]\Omega_s \\
&\quad + \frac{1}{Det(s)}\Bigg[-\varphi_\tau \frac{L}{(1+\varphi'^2(s))^{1/2}} + \frac{\delta\xi L\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})} \\
&\quad + \frac{100(\delta\xi)^{101} L\varphi'\varphi_\tau'}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\Bigg]\Omega_\xi. \qquad (5.2)
\end{aligned}$$

Recall that $\frac{\delta}{Det(s)}$ is $O(1)$ (C.12). The coefficients of Ω_s are all uniformly bounded in δ and contribute to a term of type $A_1\Omega_s$. The first coefficient of Ω_ξ we match with a term from $u\cdot\nabla q$ using the sharp-front equation (2.11). The second and third coefficients are also uniformly bounded; the second term is of the form $A_2\Omega_\xi$ and the third term $A_3\xi\Omega_\xi$.

More work is needed in classifying the terms that arise from taking $u\cdot\nabla q$ and for this purpose we revisit the derivation of (C.26) - (C.54). From (C.18) we have:

$$
\begin{aligned}
\nabla q =& \\
&\frac{\delta}{Det(s)}\mathbf{t}(s)\Omega_s && \text{(G1)}\\
&+\frac{L}{Det(s)}\mathbf{n}(s)\Omega_\xi && \text{(G2)}\\
&-\frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^{3/2}}\mathbf{n}\Omega_\xi && \text{(G3)}\\
&-\frac{\varphi'(s)^2}{(1+\varphi'^2(s))}\frac{\delta}{Det(s)}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}}+\frac{10^4(\delta\xi)^{100}}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{t}\Omega_s && \text{(G4)}\\
&-\frac{\varphi'(s)}{(1+\varphi'^2(s))}\frac{\delta}{Det(s)}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}}+\frac{10^4(\delta\xi)^{100}}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{n}\Omega_s && \text{(G5)}\\
&+\frac{\delta}{Det(s)}\frac{\varphi'\varphi'' L\xi}{(1+\varphi'^2(s))^{5/2}}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}}\right)\mathbf{t}\Omega_\xi && \text{(G6)}\\
&+\frac{\delta}{Det(s)}\frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{5/2}}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}}\right)\mathbf{n}\Omega_\xi && \text{(G7)}
\end{aligned}
$$

and u, using (C.19), is given by:

$$
\begin{aligned}
u = \iint K_\alpha^\delta \nabla^\perp q(\bar{s},\bar{\xi})Det(\bar{s})\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} =& \\
&\iint K_\alpha^\delta \delta\mathbf{n}(\bar{s})\Omega_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} && \text{(U1)}\\
&-\iint K_\alpha^\delta L\mathbf{t}(\bar{s})\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} && \text{(U2)}\\
&+\iint K_\alpha^\delta \frac{\varphi''(\bar{s})L\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\delta\mathbf{t}(\bar{s})\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} && \text{(U3)}\\
&-\iint K_\alpha^\delta \frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))}\delta\left(\frac{100(\delta\bar{\xi})^{100}}{1+100(\delta\bar{\xi})^{100}}+\frac{10^4(\delta\bar{\xi})^{100}}{(1+100(\delta\bar{\xi})^{100})^2}\right)\mathbf{n}(\bar{s})\Omega_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} && \text{(U4)}\\
&+\iint K_\alpha^\delta \frac{\varphi'(\bar{s})}{(1+\varphi'^2(\bar{s}))}\delta\left(\frac{100(\delta\bar{\xi})^{100}}{1+100(\delta\bar{\xi})^{100}}+\frac{10^4(\delta\bar{\xi})^{100}}{(1+100(\delta\bar{\xi})^{100})^2}\right)\mathbf{t}(\bar{s})\Omega_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} && \text{(U5)}
\end{aligned}
$$

$$+\iint K_\alpha^\delta \frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}} \delta \frac{\varphi'(\bar{s})\varphi''(\bar{s})L}{(1+\varphi'^2(\bar{s}))^{5/2}} \mathbf{n}(\bar{s})\Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{U6}$$

$$-\iint K_\alpha^\delta \frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}} \delta \frac{\varphi''(\bar{s})L}{(1+\varphi'^2(\bar{s}))^{5/2}} \mathbf{t}(\bar{s})\Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}, \tag{U7}$$

where we have labelled the equations above for simplicity. To find the form of the IVP it remains to study the products $G_i \cdot U_j$ for $1 \leq i, j \leq 7$.

All of the terms have previously been calculated in Appendix C; Table C.1 gives an overview of the terms that appear by counting powers of δ and ξ. The products $G_1 \cdot U, G_4 \cdot U$ and $G_5 \cdot U$ have uniformly bounded coefficients by inspection and we refer the reader to Table C.1 for the classification of these. The remaining terms need further analysis.

We first study $G_3 \cdot U$; the arguments for $G_6 \cdot U$ and $G_7 \cdot U$ follow similarly.

$$G_3 \cdot U =$$

$$-\frac{\delta^2}{Det(s)} \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}} \mathbf{n}(s)\Omega_\xi \cdot \iint K_\alpha^\delta \mathbf{n}(\bar{s})\Omega_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{5.3}$$

$$+\frac{\delta}{Det(s)} \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}} \mathbf{n}(s)\Omega_\xi \cdot \iint K_\alpha^\delta L\mathbf{t}(\bar{s})\Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{5.4}$$

$$-\frac{\delta^2}{Det(s)} \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}} \mathbf{n}(s)\Omega_\xi \cdot \iint K_\alpha^\delta \frac{\varphi''(\bar{s})L\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} \mathbf{t}(\bar{s})\Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{5.5}$$

$$+\frac{\delta^2}{Det(s)} \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}} \mathbf{n}(s)\Omega_\xi$$

$$\cdot \iint K_\alpha^\delta \frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))} \left(\frac{100(\delta\bar{\xi})^{100}}{1+100(\delta\bar{\xi})^{100}} + \frac{10^4(\delta\bar{\xi})^{100}}{(1+100(\delta\bar{\xi})^{100})^2} \right) \mathbf{n}(\bar{s})\Omega_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{5.6}$$

$$-\frac{\delta^2}{Det(s)} \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}} \mathbf{n}\Omega_\xi$$

$$\cdot \iint K_\alpha^\delta \frac{\varphi'(\bar{s})}{(1+\varphi'^2(\bar{s}))} \left(\frac{100(\delta\bar{\xi})^{100}}{1+100(\delta\bar{\xi})^{100}} + \frac{10^4(\delta\bar{\xi})^{100}}{(1+100(\delta\bar{\xi})^{100})^2} \right) \mathbf{t}(\bar{s})\Omega_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{5.7}$$

$$-\frac{\delta^2}{Det(s)} \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}} \mathbf{n}(s)\Omega_\xi \cdot \iint K_\alpha^\delta \frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}} \frac{\varphi'(\bar{s})\varphi''(\bar{s})L}{(1+\varphi'^2(\bar{s}))^{5/2}} \mathbf{n}(\bar{s})\Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{5.8}$$

$$+\frac{\delta^2}{Det(s)} \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}} \mathbf{n}(s)\Omega_\xi \cdot \iint K_\alpha^\delta \frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}} \frac{\varphi''(\bar{s})L}{(1+\varphi'^2(\bar{s}))^{5/2}} \mathbf{t}(\bar{s})\Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}, \tag{5.9}$$

for which we note that:

- All terms are uniformly bounded independent of δ.

- Terms (5.3), (5.6) and (5.7) are of the form:

$$\iint P_1 \Omega_{\bar{s}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \; \xi \Omega_{\xi}, \tag{5.10}$$

- Term (5.4) is of the form:

$$\iint P_2 \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \; \xi \Omega_{\xi}, \tag{5.11}$$

- Terms (5.5), (5.8) and (5.9) are of the form:

$$\iint P_3 \bar{\xi} \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \; \xi \Omega_{\xi}, \tag{5.12}$$

where P_i have a similar structure to that of Q_1 and Q_2 derived in §4.2. In particular, when showing the necessary Cauchy estimates on these terms that arise in the IVP the same arguments at the end of Chapter 4 hold.

It remains to study $G_2 \cdot U$. The term G_2 has a coefficient $\frac{1}{Det(s)}$ which is $O(\frac{1}{\delta})$. We show that by using the sharp front equation (2.11) and the corresponding term in (5.2) that this term is bounded uniformly in δ. We have:

$$G_2 \cdot U =$$

$$\frac{\delta}{Det(s)} \mathbf{n}(s) \Omega_{\xi} \cdot \iint K_{\alpha}^{\delta} \mathbf{n}(\bar{s}) \Omega_{\bar{s}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \tag{5.13}$$

$$-\frac{L}{Det(s)} \mathbf{n}(s) \Omega_{\xi} \cdot \iint K_{\alpha}^{\delta} L \mathbf{t}(\bar{s}) \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \tag{5.14}$$

$$+\frac{L\delta}{Det(s)} \mathbf{n}(s) \Omega_{\xi} \cdot \iint K_{\alpha}^{\delta} \frac{\varphi''(\bar{s}) L \bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} \mathbf{t}(\bar{s}) \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \tag{5.15}$$

$$-\frac{L\delta}{Det(s)} \mathbf{n}(s) \Omega_{\xi} \cdot$$

$$\iint K_{\alpha}^{\delta} \frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))} \left(\frac{100(\delta\bar{\xi})^{100}}{1+100(\delta\bar{\xi})^{100}} + \frac{10^4(\delta\bar{\xi})^{100}}{(1+100(\delta\bar{\xi})^{100})^2} \right) \mathbf{n}(\bar{s}) \Omega_{\bar{s}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \tag{5.16}$$

$$+\frac{L\delta}{Det(s)} \mathbf{n}(s) \Omega_{\xi}$$

$$\cdot \iint K_{\alpha}^{\delta} \frac{\varphi'(\bar{s})}{(1+\varphi'^2(\bar{s}))} \left(\frac{100(\delta\bar{\xi})^{100}}{1+100(\delta\bar{\xi})^{100}} + \frac{10^4(\delta\bar{\xi})^{100}}{(1+100(\delta\bar{\xi})^{100})^2} \right) \mathbf{t}(\bar{s}) \Omega_{\bar{s}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \tag{5.17}$$

$$+\frac{L\delta}{Det(s)} \mathbf{n}(s) \Omega_{\xi} \cdot \iint K_{\alpha}^{\delta} \frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}} \frac{\varphi'(\bar{s})\varphi''(\bar{s}) L}{(1+\varphi'^2(\bar{s}))^{5/2}} \mathbf{n}(\bar{s}) \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi} \tag{5.18}$$

$$-\frac{L\delta}{Det(s)} \mathbf{n}(s) \Omega_{\xi} \cdot \iint K_{\alpha}^{\delta} \frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}} \frac{\varphi''(\bar{s}) L}{(1+\varphi'^2(\bar{s}))^{5/2}} \mathbf{t}(\bar{s}) \Omega_{\bar{\xi}} \mathrm{d}\bar{s} \mathrm{d}\bar{\xi}, \tag{5.19}$$

for which:

- Terms (5.13) and (5.15) - (5.19) are uniformly bounded in δ,
- Terms (5.13), (5.16) and (5.17) are of the form:

$$\iint P_4 \Omega_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_{\xi}, \tag{5.20}$$

- Terms (5.15), (5.18) and (5.19) are of the form:

$$\iint P_5 \bar{\xi} \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_{\xi}. \tag{5.21}$$

To complete the derivation of the IVP, we now study (5.14), which writing in full is:

$$-\frac{L^2}{Det(s)} \iint \tilde{K}_{\alpha}^{\delta}(s,\bar{s},\xi,\bar{\xi}) \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi}. \tag{5.22}$$

We combine (5.22) with the first coefficient of Ω_{ξ} in (5.2):

$$\begin{aligned} &-\frac{1}{Det(s)} \frac{L}{(1+\varphi'^2(s))^{1/2}} \varphi_{\tau} \Omega_{\xi} \\ &= \frac{L^2}{Det(s)} \iint \tilde{K}_{\alpha}^{0} \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi}, \end{aligned} \tag{5.23}$$

using the sharp front equation (2.11) and the property $\int \Omega_{\xi} \mathrm{d}\xi = 1$. Using the form of $\frac{1}{Det(s)}$, we study the sum of these terms, which is of the form:

$$\begin{aligned} &\frac{1}{\delta} \iint (\tilde{K}_{\alpha}^{\delta} - \tilde{K}_{\alpha}^{0}) \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_{\xi} \\ &= \frac{1}{\delta}\Omega_{\xi} \iint (\tilde{K}_{\alpha}^{\delta} - \tilde{K}_{\alpha}^{0}) \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \\ &= \xi \frac{1}{\delta\xi}\Omega_{\xi} \iint (\tilde{K}_{\alpha}^{\delta} - \tilde{K}_{\alpha}^{0}) \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}. \end{aligned} \tag{5.24}$$

Rewriting the equation in this form we can utilise Lemma 4.6 and Lemma 4.7. That is for $\delta\xi \to 0$ this term is negligible, else this term is $O(1)$. We may therefore assume that (5.15) contributes a term of the form:

$$\iint P_6\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \xi\Omega_\xi. \tag{5.25}$$

Using the details shown above, and summarised in Table C.1, we obtain the IVP of interest:

$$\begin{cases} \Omega_\tau + F(\tau,\Omega) = 0 \\ \Omega|_{\tau=0} = \Omega_0 \end{cases} \tag{5.26}$$

where:

$$\begin{aligned} F(\tau,\Omega) = A_1\Omega_s + A_2\Omega_\xi + A_3\xi\Omega_\xi + B_1\iint P_1\Omega_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \xi\Omega_\xi \\ +B_2\iint P_2\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \xi\Omega_\xi + B_3\iint P_3\bar{\xi}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \xi\Omega_\xi + B_4\iint P_4\Omega_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_\xi \\ +B_5\iint P_5\bar{\xi}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\Omega_\xi + B_6\iint P_6\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \xi\Omega_\xi + B_7\iint P_7\Omega_{\bar{s}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_s \\ +B_8\iint P_8\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_s + B_9\iint P_9\bar{\xi}\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\ \Omega_s. \end{aligned} \tag{5.27}$$

For each i the terms A_i, B_i and P_i can be explicitly constructed, although that is not necessary here. Note that we have shown that all of the terms P_i are bounded independently of δ and so we achieve existence of solutions in short-time, with the time interval of existence independent of δ. In addition, the terms P_i are of the form Q_i as in §4.2. This allows us to dispose of most of the proof by the same arguments presented in Chapter 4. The remainder of this chapter contains the precise form of the existence result and the steps required to complete the proof.

5.2 Function Spaces

We first define the function spaces that are required for applying the ACK theorem to the IVP derived in the previous section (5.26). As in Chapter 4 we only complexify the space variables, we have $s, \xi \in \mathbb{C}$ and $\tau \in \mathbb{R}$. The Cauchy estimates, as required for an existence theorem, would fail for terms of the form $\xi\Omega_\xi$ which appear here. Such terms require a domain that grows with $\Im\xi$. For this purpose we consider functions that are analytic and L^2 in the space variables in the "strip" $D(\rho)$ (as in §4.4.1) and the "bow-tie" $\Sigma(m)$ defined as follows:

$$D(\rho) = \mathbb{R} \times (-\rho, \rho) = \{s \in \mathbb{C} : \Im x \in (-\rho, \rho)\}, \tag{5.28}$$

$$\begin{aligned}\Sigma(m) &= \{\xi \in \mathbb{C} : \text{for} \quad \theta = \tan^{-1}(m), \quad 0 \leq |\Re\xi| \leq \cos\theta \quad \text{and} \quad |\Im\xi| \leq \sin\theta\} \\ &\cup \{\xi \in \mathbb{C} : \text{for} \quad \theta = \tan^{-1}(m) \quad |\Re\xi| \geq \cos\theta \quad \text{and} \quad |\Im\xi| \leq m|\Re\xi|\},\end{aligned} \tag{5.29}$$

and the corresponding paths along which the L^2 integration is performed:

$$\Gamma(b) = \{s \in \mathbb{C} : \Im s = b\}, \tag{5.30}$$

$$\begin{aligned}\gamma(m') &= \{\xi \in \mathbb{C} : \text{for} \quad \theta' = \tan^{-1}(m'), \quad 0 \leq |\Re\xi| \leq \cos\theta' \quad \text{and} \quad |\Im\xi| = \sin\theta'\} \\ &\cup \{\xi \in \mathbb{C} : \text{for} \quad \theta' = \tan^{-1}(m') \quad |\Re\xi| \geq \cos\theta' \quad \text{and} \quad |\Im\xi| = m'|\Re\xi|\} \\ &= \gamma_1(m') \cup \gamma_2(m').\end{aligned} \tag{5.31}$$

The bow-tie details are illustrated below for $\xi = x + iy$:

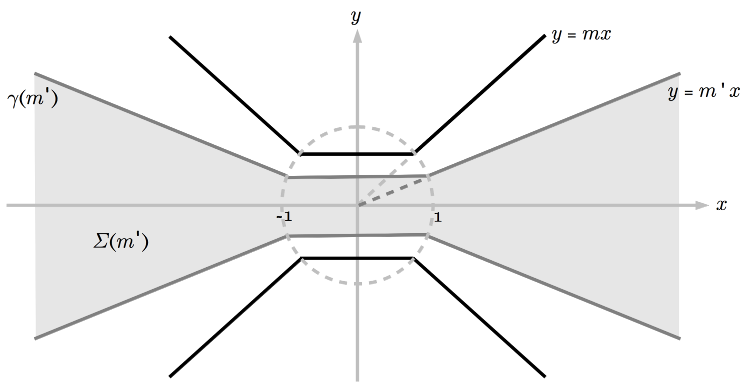

Figure 5.1: Bow-Tie Domain - $\Sigma(m')$ and $\gamma(m')$

Recall the following from Chapter 4:

Definition 5.1. *Given $l \in \mathbb{N}$ and $\rho > 0$, $H^{l,\rho}$ is the set of all complex functions $f(s)$ such that:*

- *f is analytic in $D(\rho)$ and periodic in $\Re s$.*
- *$\partial_s^\alpha f \in L^2(\Gamma(\Im s))$ for $|\Im s| < \rho$, $\alpha \leq l$; i.e. if $|\Im s| < \rho$, then $\partial_s^\alpha f(\Re s + i\Im s)$ is a square integrable function of $\Re s$.*

- *The norm $\|f\|_{l,\rho}$ is finite, where:*

$$\|f\|_{l,\rho} = \sum_{\alpha=0}^{l} \|\partial_s^\alpha f\|_\rho. \tag{5.32}$$

We now define a function space where the functions are analytic in a strip in the first variable and in the bow-tie for the second variable.

Definition 5.2. *Given $l \in \mathbb{N}$, $\rho > 0$, $0 < m \leq 1$, $K^{l,\rho,m}$ is the set of all complex functions $f(s,\xi)$ such that:*

- *f is analytic in $D(\rho) \times \Sigma(m)$ and periodic in both $\Re s$ and $\Re\xi$.*
- *$\partial_\xi^{\alpha_1}\partial_s^{\alpha_2} f \in L^2(\gamma(m'); H^{0,\rho})$ with $|m'| \leq m$, $\alpha_1 + \alpha_2 \leq l$.*
- *The norm $\|f\|_{1,l,\rho,m}$ is finite, where:*

$$\|f\|_{1,l,\rho,m} = \sum_{\alpha_1+\alpha_2 \leq l} \sup_{|m'|<m} \left\| \|\partial_\xi^{\alpha_1}\partial_s^{\alpha_2} f(\cdot,\xi)\|_{0,\rho} \right\|_{L^2(\gamma(m'))}. \tag{5.33}$$

Motivation for the restriction of the parameter $0 < m \leq 1$ will be outlined in the next section. This corresponds to the definition in Sammartino and Caflisch (1998a) on the angle of the slope $0 < \theta \leq \frac{\pi}{4}$. We introduce the norm $\|f\|_{1,l,\rho,m}$ with the 1 to distinguish this from the norm also dependent on three variables from the previous chapter.

In the next section we show that $\xi\Omega_\xi$ satisfies the required Cauchy estimates; for simplicity we now define a space for functions of one variable analytic on $\Sigma(m)$.

Definition 5.3. *For $f(\xi)$ analytic on the bow-tie $\Sigma(m)$, for some $m > 0$, the norm $\|f\|_{1,m}$ is given by:*

$$\|f\|_{1,m} = \sup_{|m'|\leq m} \|f(\cdot + iy)\|_{L^2(\gamma(m'))}. \tag{5.34}$$

Definition 5.4. *Given $l \in \mathbb{N}$ and $m > 0$, $H^{1,l,m}$ is the set of all complex functions $f(\xi)$ such that:*

- *f is analytic in $\Sigma(m)$ and periodic in $\Re s$.*
- *$\partial_s^\alpha f \in L^2(\gamma(m'))$ for $|m'| < m$, $\alpha \leq l$.*
- *The norm $\|f\|_{1,l,\rho}$ is finite, where:*

$$\|f\|_{1,l,m} = \sum_{\alpha=0}^{l} \|\partial_\xi^\alpha f\|_{1,m}. \tag{5.35}$$

It remains to define the function spaces that depend on time:

Definition 5.5. *Given $l \in \mathbb{N}$, $\rho, \beta, T > 0$ and $0 < m \leq 1$, the function $f(s, \xi, t)$ is in $K^{l,\rho,m}_{\beta,T}$ if and only if*

- *$f(s, \xi, t)$ is periodic in $\Re s$ and $\Re \xi$ and analytic in $D(\rho) \times \Sigma(m)$.*
- *$\partial_t^k f \in C([0, T]; K^{l-k,\rho,m})$ for $0 \leq k \leq l$.*
- *The norm $\|f\|_{1,l,\rho,m,\beta,T}$ is finite, where:*

$$\|f\|_{1,l,\rho,m,\beta,T} = \sum_{k=0}^{l} \sup_{0 \leq t \leq T} \|\partial_t^k f(\cdot, \cdot, t)\|_{1,l-k,\rho-\beta t,m-\beta t}. \tag{5.36}$$

5.3 Existence Result

We prove the following:

Theorem 5.6 (Existence of Exact Solutions)**.** *Let $\Omega_0 \in K^{l,\rho,m}$, $l \geq 4$ and $0 < m \leq 1$. Then equation (5.26) has a unique solution $\Omega \in K^{l,\rho_0,m_0}_{\beta_0,T}$ for some $0 < \rho_0 < \rho, 0 < m_0 < m, \beta_0 > 0$ and $T > 0$. In particular, T is independent of δ.*

In Chapter 4, the necessary Cauchy estimates were proved using lemmas from Sammartino and Caflisch (1998a). We state the analagous lemmas for the function spaces introduced in §5.2.

Lemma 5.7. *Let $f \in H^{1,l,m}$. For $0 < m' < m$ then:*

$$\|\partial_\xi f\|_{1,l,m'} \leq \frac{\|f\|_{1,l,m}}{m - m'}. \tag{5.37}$$

Lemma 5.8. *Let $f(s, \xi) \in K^{l,\rho,m}$ with $l \geq 4$ and let $0 < \rho' < \rho$, then:*

$$\|f_s\|_{1,l,\rho',m} \leq \frac{\|f\|_{1,l,\rho,m}}{\rho - \rho'}. \tag{5.38}$$

Lemma 5.9. *Let $f, g \in K^{l,\rho,m}_{\beta,T}$ and $l \geq 4$. Then $f \cdot g \in K^{l,\rho,m}_{\beta,T}$, and*

$$\|f \cdot g\|_{1,l,\rho,m,\beta,T} \leq c\|f\|_{1,l,\rho,m,\beta,T}\|g\|_{1,l,\rho,m,\beta,T}. \tag{5.39}$$

Lemma 5.10. *Let $f, g \in K^{l,\rho,m}$ with $l \geq 4$, and $0 < \rho' < \rho$. Then:*

$$\|g\partial_s f\|_{1,l,\rho',m} \leq \|g\|_{1,l,\rho',m} \frac{\|f\|_{1,l,\rho,m}}{\rho - \rho'}. \tag{5.40}$$

Lemma 5.11. *Let $f, g \in K^{l,\rho,m}$ with $l \geq 4$, and $0 < m' < m$. Then:*

$$\|g\partial_\xi f\|_{1,l,\rho,m'} \leq \|g\|_{1,l,\rho,m'} \frac{\|f\|_{1,l,\rho,m}}{m - m'}. \tag{5.41}$$

Remark 5.12. *The preceding results (Lemmas 5.7 - 5.11) appear in Sammartino and Caflisch (1998a) in the case of the upper half plane. The proofs for $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ are completely analogous. As seen in the proof presented on the strip before (page 76), all the proofs of this type of Cauchy estimate are local and do not depend on the overall geometry of the domain.*

In order to prove the existence result, given the IVP derived in (5.26) we rewrite this in the new variable $\Omega = \Omega_0 + W$ where W is posed on a Banach Space and use the ACK theorem to show the existence for solutions to:

$$\begin{cases} W_t + G(t, W) = 0 \\ \quad \Omega|_{t=0} = \Omega_0 \end{cases} \tag{5.42}$$

where again we have replaced the time variable by t for ease of notation. We refer the reader to Chapter 4 for a discussion of this; we do not rewrite the equation here, but note that due to the form of the IVP in (5.26) it is clear that the only difference in the proofs of Theorem 5.6 and Theorem 4.18 is that we now require Cauchy estimates on terms of the form $\xi\Omega_\xi$. The time independence has been discussed in §5.1.

5.3.1 Cauchy Estimates on $\xi\Omega_\xi$

For the domain as pictured in Figure 5.2 we prove the following results:

Lemma 5.13. *For $f(\xi)$ analytic in the bow-tie $\Sigma(m)$ for $m > 0$:*

$$\|\xi\partial_\xi f\|_{m'} \leq C \frac{\|f\|_m}{m - m'}. \tag{5.43}$$

holds for $0 < m' < m$, and C constant.

Lemma 5.14. *Let $f(\xi) \in H^{1,l,m}$ with $l \geq 4$ and let $0 < m' < m \leq 1$, then:*

$$\|\xi f_\xi\|_{1,l,m'} \leq \frac{\|f\|_{1,l,m}}{m - m'}. \tag{5.44}$$

of which the latter gives the restriction on the values of m.

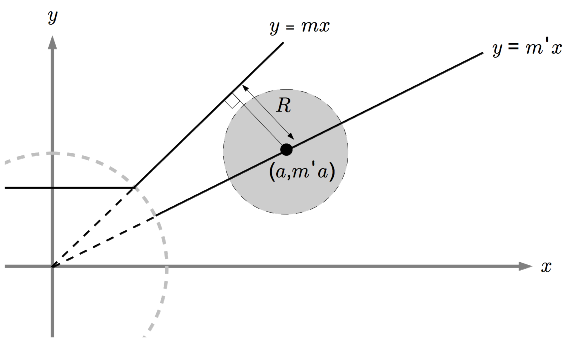

Figure 5.2: Cauchy Estimates

Proof of Lemma 5.13 It remains to prove this for points in $\Sigma(m)$ outside of the unit ball. Inside the unit ball ξ is bounded and so the estimates follow from Chapter 4.

Outside of the unit ball consider a point on the line $y = m'x$, that is $\xi = a+im'a$. Using standard results for the distance between a point and a line, we have that for fixed m, the maximal radius of a ball centred at this point and contained in the domain $\Sigma(m)$ is:

$$R = \frac{|m'a - ma|}{\sqrt{1+m^2}} = \frac{a(m-m')}{\sqrt{1+m^2}}$$

as pictured.

We choose $r = \frac{a(m-m')}{2\sqrt{1+m^2}}$. Setting $\xi = a + ib$ and using the Cauchy integral formula we have:

$$\xi f_\xi = \frac{(a+ib)}{2\pi i} \int_0^{2\pi} \frac{f(a+ib+re^{it})}{r^2 e^{i2t}} re^{it} \mathrm{d}t. \tag{5.45}$$

Evaluating this at a point outside the unit ball and along the line $y = \epsilon x$ with $\epsilon < m'$ we have:

$$(a+i\epsilon a) f_\xi(a+i\epsilon a) = \frac{(a+i\epsilon a)}{2\pi i} \int_0^{2\pi} \frac{f(a+i\epsilon a+re^{it})}{re^{i2t}} \mathrm{d}t$$

$$= \frac{(1+i\epsilon)\sqrt{1+m^2}}{2\pi i(m-m')} \int_0^{2\pi} \frac{f(a+i\epsilon a+re^{it})}{e^{i2t}} \mathrm{d}t \tag{5.46}$$

giving, for $\epsilon < m' < m$, $|1+i\epsilon| \le \sqrt{1+m^2}$:

$$|(a+i\epsilon a)f_\xi(a+i\epsilon a)|^2 \le \frac{2\pi(1+m^2)^2}{4\pi^2(m-m')^2} \int_0^{2\pi} |f(a+i\epsilon a+re^{it})|^2 \mathrm{d}t$$

and so:

$$\begin{aligned} &\int_{\gamma_2(\epsilon)} |(a+i\epsilon a)f_\xi(a+i\epsilon a)|^2 \mathrm{d}a \\ &\le \frac{(1+m^2)^2}{2\pi(m-m')^2} \int_0^{2\pi} \left(\int_{\gamma_2(\epsilon)} |f(a+i\epsilon a+re^{it})|^2 \mathrm{d}a \right) \mathrm{d}t. \end{aligned} \tag{5.47}$$

The inner integral in the RHS of (5.47) is precisely $\|f\|_{1,m}^2$ as, for fixed t, $a+i\epsilon a+re^{it}$ decribes a line with gradient less than m by construction. Taking the supremum over $|\epsilon| \le m'$ in the LHS of (5.47), and taking square roots, we obtain:

$$\|\xi f_\xi\|_{1,m'} \le \frac{1+m^2}{m-m'} \|f\|_{1,m} \tag{5.48}$$

as required. $\square$

Proof of Lemma 5.14 We have, for $f \in H^{1,l,m}$ the following:

$$\begin{aligned} \|\xi f_\xi\|_{1,l,m} &= \sum_{\alpha=0}^{l} \|\partial_\xi^\alpha(\xi f_\xi)\|_{1,m'} \\ &\le \sum_{\alpha=0}^{l} \|\xi \partial_\xi^\alpha f_\xi\|_{1,m'} + \|\partial_\xi^\alpha f\|_{1,m'} \end{aligned} \tag{5.49}$$

using the Leibniz rule. By Definition 5.4 this gives:

$$\begin{aligned} (5.49) &= \|f\|_{1,m'} + \sum_{\alpha=0}^{l} \|\xi \partial_\xi^\alpha f_\xi\|_{1,m'} \\ &= \|f\|_{1,m'} + \sum_{\alpha=0}^{l} \|\xi (\partial_\xi^\alpha f)_\xi\|_{1,m'} \end{aligned} \tag{5.50}$$

$$\leq \|f\|_{1,m} + \sum_{\alpha=0}^{l} \frac{C}{m-m'} \|\partial_\xi^\alpha f\|_{1,m} \tag{5.51}$$

$$= \|f\|_{1,m} + \frac{C}{m-m'} \|f\|_{1,m}, \tag{5.52}$$

where (5.51) follows by the properties of a Banach scale and Lemma 5.13. We note that we obtain precisely the estimate as needed if we make the assumption $\frac{1}{m-m'} \leq 1$. That is $0 < m - m' \leq 1$, which holds by the restriction we assumed on the values of m. $\square$

We have proved the remaining details needed for existence which, alongside the previously shown properties in Chapter 4, completes the proof of Theorem 5.6.

5.4 Discussion

An open question for the SQG equation, and in turn giving an insight into the three-dimensional Euler equation, is the existence of smooth almost-sharp front solutions. We have shown in Chapter 3, that even for the simpler case $\alpha < 1$ it has not been possible as yet to prove such existence. This is linked to the open well-posedness problem for the Prandtl equations, see for example Grenier (2004).

The existence results presented here and in Chapter 4 (Theorems 4.18 and 5.6) ensure that for suitable initial conditions, there exist both approximate solutions and exact solutions to the α-equation when posed on the two-dimensional cylinder, in the analytic case. These solutions are not however almost-sharp fronts, they are asymptotic to such solutions by the construction. The interest in studying such solutions is in the comparison to the case when $\alpha = 1$, and in introducing a method that may be extended to prove existence of the same type of solution for the SQG equation. Unfortunately due to the presence of a logarithmic term, $\xi \log \xi \Omega_\xi$, these methods do not directly extend; the Cauchy estimates as required by the ACK theorem will not hold. These are however new results for the α-equation; the existence of exact solutions in the SQG case still remains open.

In the next chapter we continue to study the existence of solutions to the α-equation, and can in fact improve on the results of these two chapters by showing that there exist analytic solutions taking the form of an almost-sharp front.

Chapter 6

Analytic Almost-Sharp Fronts

In Chapters 4 and 5 we have shown that for $0 < \alpha < 1$ there exist both approximate and exact solutions to the α-equation, which by construction are of a form asymptotic to the almost-sharp fronts defined in Chapter 2. The interest for the SQG equation is in constructing solutions that are constant above and below two curves; solutions that are almost-sharp fronts.

The result contained within this chapter gives existence and uniqueness of analytic solutions to the α-equation for $\alpha < 1$. As described in Chapter 5, the hope would be to extend the methods presented to the case when $\alpha = 1$. For the methods previously introduced, the growth of the terms that appear in the equations are of order ξ; the corresponding terms in the SQG equation grow as $\xi \log \xi$ for which we cannot obtain Cauchy estimates.

In this chapter we introduce a new method for constructing almost-sharp front solutions to the α-equation. This construction has been introduced by Jose Rodrigo in current work. We study the evolution of the boundaries of the connected regions $\Omega = \frac{1}{2}$ and $\Omega = -\frac{1}{2}$ as shown in Figure 6.1. Note that we define these in such a way so that there is no ambiguity - Ω could take these values inside the shaded region A, which is why we specify the connected components. As with the previous cases we are interested in studying the evolution of region A. Note that, as defined above, the curves f and g are level sets and so will be advected by the fluid. We use this property in deriving the existence result. In the next section we construct a family of almost-sharp fronts, Ω, dependent on the parameter δ.

The notation within this chapter and the definitions of objects such as the curve φ (for some φ sitting inside A), or the form of an almost sharp front q (2.15), remain the same as used previously. We also assume that φ continues to satisfy the sharp front equation (2.11):

$$\frac{\partial \varphi}{\partial t}(x,t) = \int\limits_{\mathbb{R}/\pi\mathbb{Z}} \frac{\frac{\partial \varphi}{\partial x}(x,t) - \frac{\partial \varphi}{\partial \bar{x}}(\bar{x},t)}{(\cosh(\varphi(x,t) - \varphi(\bar{x},t)) - \cos(x-\bar{x}))^{\alpha/2}} \mathrm{d}\bar{x}.$$

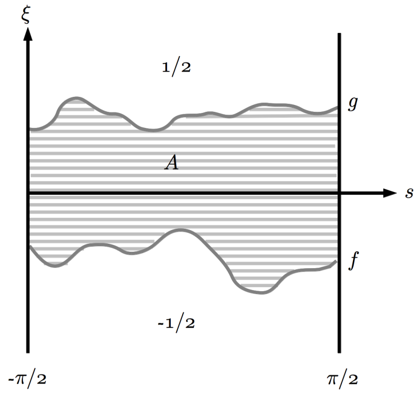

Figure 6.1: Boundaries of connected regions

6.1 Change of Coordinates

We wish to study a family of almost sharp-front solutions on a fixed domain (see also Chapters 3 and 4), for which we introduce the following change of coordinates that describes the region A in new variables $(s,\zeta) \in [0,1) \times [-1,1]$. We first define the mapping R_1 as shown in Figure 6.2 as:

$$\xi = \frac{g(s) - f(s)}{2}\zeta + \frac{g(s) + f(s)}{2} \tag{6.1}$$

so that $\zeta = \pm 1$ correspond to the top and bottom curves g and f respectively.

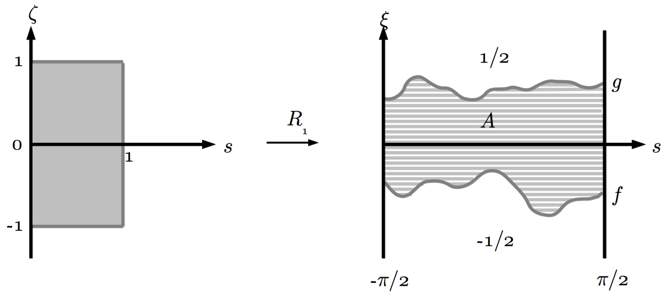

Figure 6.2: R_1

Recalling the notation from Chapter 3 for the renormalised arc length R we construct a mapping R_2 as in (3.5) and illustrated below:

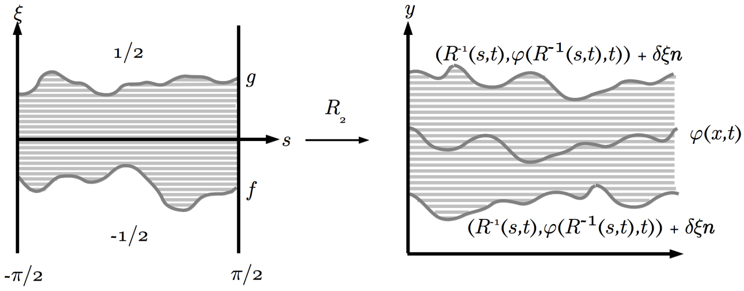

Figure 6.3: R_2

We define a map from the bounded domain $(s,\zeta) \in \mathbb{R}/\pi\mathbb{Z} \times [-1,1]$ using $R_2 \circ R_1$:

$$
\begin{aligned}
(x,y,t) = {} & (R^{-1}(s,\tau), \varphi(R^{-1}(s,\tau),\tau),\tau) + \\
& \delta\left(\frac{g(R^{-1}(s,\tau),\tau) - f(R^{-1}(s,\tau),\tau)}{2}\zeta + \frac{g(R^{-1}(s,\tau),\tau) + f(R^{-1}(s,\tau),\tau)}{2}\right)\mathbf{n}
\end{aligned} \tag{6.2}
$$

where n is defined to be the unit normal of φ at a point $R^{-1}(s,\tau)$, that is:

$$\mathbf{n}(R^{-1}(s,\tau)) = \frac{(-\varphi'(R^{-1}(s,\tau),\tau),1,0)}{(1+\varphi'^2(R^{-1}(s,\tau),\tau))^{1/2}}.$$

We study a family of almost-sharp fronts, by construction, $q(x,y,t) = \Omega(s,\zeta,\tau)$, indexed by a parameter $\delta > 0$. As in the previous chapters we simplify the notation by writing, for example $f(s) = f(R^{-1}(s,\tau),\tau)$, where it is clear. We then have:

$$x = R^{-1}(s,\tau) - \delta\left(\frac{g(s)-f(s)}{2}\zeta + \frac{g(s)+f(s)}{2}\right)\frac{\varphi'(s)}{(1+\varphi'^2(s))^{1/2}}, \tag{6.3}$$
$$y = \varphi(s) + \delta\left(\frac{g(s)-f(s)}{2}\zeta + \frac{g(s)+f(s)}{2}\right)\frac{1}{(1+\varphi'^2(s))^{1/2}}. \tag{6.4}$$

Remark 6.1. *We note that comparing to the smooth case presented in Chapter 3, the functions f and g represent the distance in the normal direction of the curve φ from the bottom and top boundaries, playing the same role as ξ previously.*

We now rewrite the α-equation, (2.1)-(2.2), in the new coordinate system. To do this we first calculate the derivatives:

$$\begin{aligned}
\frac{\partial x}{\partial s} &= R_s^{-1} + \delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\left(-\frac{\varphi'' R_s^{-1}}{(1+\varphi'^2)^{3/2}}\right) \\
&\quad + \delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)R_s^{-1}\left(-\frac{\varphi'}{(1+\varphi'^2)^{1/2}}\right),
\end{aligned} \tag{6.5}$$
$$\begin{aligned}
\frac{\partial y}{\partial s} &= \varphi' R_s^{-1} - \delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'\varphi'' R_s^{-1}}{(1+\varphi'^2)^{3/2}} \\
&\quad + \delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)R_s^{-1}\frac{1}{(1+\varphi'^2)^{1/2}},
\end{aligned} \tag{6.6}$$
$$\frac{\partial x}{\partial \zeta} = -\delta\left(\frac{g-f}{2}\right)\frac{\varphi'}{(1+\varphi'^2)^{1/2}}, \tag{6.7}$$
$$\frac{\partial y}{\partial \zeta} = \delta\left(\frac{g-f}{2}\right)\frac{1}{(1+\varphi'^2)^{1/2}}, \tag{6.8}$$
$$\begin{aligned}
\frac{\partial x}{\partial \tau} &= R_t^{-1} + \delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\left(\frac{-\varphi'' R_t^{-1} - \varphi'_t}{(1+\varphi'^2)^{3/2}}\right) \\
&\quad - \delta\left(\frac{g'R_\tau^{-1} + g_\tau - f'R_\tau^{-1} - f_\tau}{2}\zeta - \frac{g'R_\tau^{-1} + g_\tau + f'R_\tau^{-1} + f_\tau}{2}\right)\frac{\varphi'}{(1+\varphi'^2)^{1/2}},
\end{aligned} \tag{6.9}$$

$$\frac{\partial y}{\partial \tau} = \varphi' R_\tau^{-1} + \varphi_\tau + \delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\left(\frac{-\varphi'[\varphi'' R_\tau^{-1} + \varphi_\tau']}{(1+\varphi'^2)^{3/2}}\right)$$
$$+ \delta\left(\frac{g' R_\tau^{-1} + g_\tau - f' R_\tau^{-1} - f_\tau}{2}\zeta - \frac{g' R_\tau^{-1} + g_\tau + f' R_\tau^{-1} + f_\tau}{2}\right)\frac{1}{(1+\varphi'^2)^{1/2}}. \tag{6.10}$$

As previously we also have:

$$\partial_x = \frac{1}{Det(s)}\frac{\partial y}{\partial \zeta}\partial_s - \frac{1}{Det(s)}\frac{\partial y}{\partial s}\partial_\zeta, \quad \partial_y = -\frac{1}{Det(s)}\frac{\partial x}{\partial \zeta}\partial_s + \frac{1}{Det(s)}\frac{\partial x}{\partial s}\partial_\zeta,$$
$$\partial_t = \frac{I}{Det(s)}\partial_s + \frac{II}{Det(s)}\partial_\zeta + \partial_\tau$$

and:

$$Det(s) = \frac{\partial x}{\partial s}\frac{\partial y}{\partial \zeta} - \frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial s}, \quad I = \frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \zeta}, \quad II = \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial \tau}.$$

The determinant is as follows:

$$\begin{aligned}
Det(s) &= \frac{\partial x}{\partial s}\frac{\partial y}{\partial \zeta} - \frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial s} \\
&= \delta\frac{g-f}{2}R_s^{-1}\frac{1}{(1+\varphi^2)^{1/2}} + \delta^2\frac{g-f}{2}\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\left(\frac{-\varphi'' R_s^{-1}}{(1+\varphi^2)^2}\right) \\
&\quad -\delta^2\frac{g-f}{2}\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{R_s^{-1}\varphi'}{(1+\varphi'^2)} \\
&\quad +\delta\frac{g-f}{2}\frac{\varphi'^2 R_s^{-1}}{(1+\varphi'^2)^{1/2}} - \delta^2\frac{g-f}{2}\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'^2\varphi'' R_s^{-1}}{(1+\varphi'^2)^2} \\
&\quad +\delta^2\frac{g-f}{2}\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{R_s^{-1}\varphi'}{(1+\varphi'^2)} \\
&= \delta\frac{g-f}{2}R_s^{-1}(1+\varphi'^2)^{1/2} - \delta^2\frac{g-f}{2}\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'' R_s^{-1}}{(1+\varphi'^2)} \\
&= L\delta\frac{g-f}{2} - L\delta^2\frac{g-f}{2}\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi''}{(1+\varphi'^2)^{3/2}} \\
&= L\delta\frac{g-f}{2}\left[1 - \delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi''}{(1+\varphi'^2)^{3/2}}\right]
\end{aligned} \tag{6.11}$$

which follows by recalling that $R_s^{-1} = \frac{L}{(1+\varphi'^2)^{1/2}}$, and again is $O(\delta)$. For I we find:

$$
\begin{aligned}
I &= \frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \zeta} \\
&= -\delta \frac{g-f}{2}\frac{\varphi'^2 R_\tau^{-1}}{(1+\varphi'^2)^{1/2}} - \delta \frac{g-f}{2}\frac{\varphi'\varphi_\tau}{(1+\varphi'^2)^{1/2}} \\
&\quad +\delta^2 \frac{g-f}{2}\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'^2[\varphi'' R_\tau^{-1} + \varphi_\tau]}{(1+\varphi'^2)^2} \\
&\quad -\delta^2 \frac{g-f}{2}\left(\frac{g' R_\tau^{-1} + g_\tau - f' R_\tau^{-1} - f_\tau}{2}\zeta - \frac{g' R_\tau^{-1} + g_\tau + f' R_\tau^{-1} + f_\tau}{2}\right)\frac{\varphi'}{(1+\varphi'^2)} \\
&\quad -\delta \frac{g-f}{2}\frac{R_\tau^{-1}}{(1+\varphi'^2)^{1/2}} + \delta^2 \frac{g-f}{2}\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'' R_\tau^{-1} + \varphi'_\tau}{(1+\varphi'^2)^2} \\
&\quad +\delta^2 \frac{g-f}{2}\left(\frac{g' R_\tau^{-1} + g_\tau - f' R_\tau^{-1} - f_\tau}{2}\zeta - \frac{g' R_\tau^{-1} + g_\tau + f' R_\tau^{-1} + f_\tau}{2}\right)\frac{\varphi'}{(1+\varphi'^2)} \\
&= -\delta \frac{g-f}{2}(1+\varphi'^2)^{1/2} R_\tau^{-1} - \delta \frac{g-f}{2}\frac{\varphi'\varphi_\tau}{(1+\varphi'^2)^{1/2}} \\
&\quad +\delta^2 \frac{g-f}{2}\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'' R_\tau^{-1} + \varphi'_\tau}{(1+\varphi'^2)},
\end{aligned}
\tag{6.12}
$$

and for II:

$$
\begin{aligned}
II &= \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial \tau} \\
&= \varphi' R_s^{-1} R_\tau^{-1} - R_s^{-1}\varphi' R_\tau^{-1} - \varphi_\tau R_s^{-1} \\
&\quad -\delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'\varphi'' R_s^{-1} R_\tau^{-1}}{(1+\varphi'^2)^{3/2}} + \delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{R_s^{-1} R_\tau^{-1}}{(1+\varphi'^2)^{1/2}} \\
&\quad +\delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi' R_s^{-1}[\varphi'' R_\tau^{-1} + \varphi'_\tau]}{(1+\varphi'^2)^{3/2}} \\
&\quad -\delta\left(\frac{g' R_\tau^{-1} + g_\tau - f' R_\tau^{-1} - f_\tau}{2}\zeta - \frac{g' R_\tau^{-1} + g_\tau + f' R_\tau^{-1} + f_\tau}{2}\right)\frac{R_s^{-1}}{(1+\varphi'^2)^{1/2}} \\
&\quad -\delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi' R_s^{-1}[\varphi'' R_t^{-1} + \varphi'_t]}{(1+\varphi'^2)^{3/2}} \\
&\quad -\delta\left(\frac{g' R_\tau^{-1} + g_\tau - f' R_\tau^{-1} - f_\tau}{2}\zeta - \frac{g' R_\tau^{-1} + g_\tau + f' R_\tau^{-1} + f_\tau}{2}\right)\frac{\varphi'^2 R_s^{-1}}{(1+\varphi'^2)^{1/2}} \\
&\quad +\delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'' R_s^{-1}\varphi' R_\tau^{-1}}{(1+\varphi'^2)^{3/2}} + \delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'' R_s^{-1}\varphi_\tau}{(1+\varphi'^2)^{3/2}} \\
&\quad +\delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{\varphi'^2 R_s^{-1} R_\tau^{-1}}{(1+\varphi'^2)^{1/2}} + \delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{\varphi'\varphi_\tau R_s^{-1}}{(1+\varphi'^2)^{1/2}} \\
&\quad +O(\delta^2)
\end{aligned}
$$

$$
\begin{aligned}
&= -\varphi_\tau R_s^{-1} + \delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi''\varphi_\tau R_s^{-1}}{(1+\varphi'^2)^{3/2}} \\
&+\delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{\varphi'\varphi_\tau R_s^{-1}}{(1+\varphi'^2)^{1/2}} \\
&-\delta\left(\frac{g'R_\tau^{-1}+g_\tau-f'R_\tau^{-1}-f_\tau}{2}\zeta - \frac{g'R_\tau^{-1}+g_\tau+f'R_\tau^{-1}+f_\tau}{2}\right)(1+\varphi'^2)^{1/2}R_s^{-1} \\
&+\delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{R_s^{-1}R_\tau^{-1}}{(1+\varphi'^2)^{1/2}} + O(\delta^2) \\
&= -\frac{\varphi_\tau L}{(1+\varphi'^2)^{1/2}} + \delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi''\varphi_\tau L}{(1+\varphi'^2)^2} \\
&+\delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{\varphi'\varphi_\tau L}{(1+\varphi'^2)} \\
&-\delta L\left(\frac{g'R_\tau^{-1}+g_\tau-f'R_\tau^{-1}-f_\tau}{2}\zeta - \frac{g'R_\tau^{-1}+g_\tau+f'R_\tau^{-1}+f_\tau}{2}\right) \\
&+\delta\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{LR_\tau^{-1}}{(1+\varphi'^2)} + O(\delta^2). \qquad (6.13)
\end{aligned}
$$

For clarity we omit the calculations, however the $O(\delta^2)$ term is precisely:

$$
\begin{aligned}
&\delta^2\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\times \\
&\left(\frac{g'R_\tau^{-1}+g_\tau-f'R_\tau^{-1}-f_\tau}{2}\zeta - \frac{g'R_\tau^{-1}+g_\tau+f'R_\tau^{-1}+f_\tau}{2}\right)\frac{\varphi''L}{(1+\varphi'^2)^{3/2}} \\
&-\delta^2\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\left(\frac{g'-f'}{2}\zeta + \frac{g'+f'}{2}\right)\frac{L[\varphi''R_\tau^{-1}+\varphi_\tau^{-1}]}{(1+\varphi'^2)^{3/2}}.
\end{aligned}
$$

Remark 6.2. *Note that* $\frac{I}{Det(s)}$ *is* $O(1)$, *yet* $\frac{II}{Det(s)}$ *contains a term of size* $O(\frac{1}{\delta})$. *Such a term could cause the time of existence to shrink with* δ. *We are able to prove the existence of solutions for time independent of* δ *as we show that this term does not appear in the final IVP by a suitable method to be introduced in §6.3.*

We now calculate the velocity, we have:

$$
\begin{aligned}
\frac{\partial q}{\partial x} &= \frac{1}{Det(s)}\frac{\partial y}{\partial \zeta}\Omega_s - \frac{1}{Det(s)}\frac{\partial y}{\partial s}\Omega_\zeta \\
&= \frac{1}{Det(s)}\delta\frac{g-f}{2}\frac{1}{(1+\varphi'^2)^{1/2}}\Omega_s - \frac{1}{Det(s)}\frac{\varphi' L}{(1+\varphi'^2)^{1/2}}\Omega_\zeta \\
&+\frac{1}{Det(s)}\delta\left(\frac{g-f}{2}\zeta + \frac{g+f}{2}\right)\frac{\varphi'\varphi'' L}{(1+\varphi'^2)^2}\Omega_\zeta
\end{aligned}
$$

$$-\frac{1}{Det(s)}\delta\bigg(\frac{g'-f'}{2}\zeta+\frac{g'+f'}{2}\bigg)\frac{L}{(1+\varphi'^2)}\Omega_\zeta \tag{6.14}$$

$$\frac{\partial q}{\partial y}=\frac{1}{Det(s)}\frac{\partial x}{\partial s}\Omega_\zeta-\frac{1}{Det(s)}\frac{\partial x}{\partial \zeta}\Omega_s$$

$$=\frac{1}{Det(s)}\delta\frac{g-f}{2}\frac{\varphi'}{(1+\varphi'^2)^{1/2}}\Omega_s+\frac{1}{Det(s)}\frac{L}{(1+\varphi'^2)^{1/2}}\Omega_\zeta$$

$$-\frac{1}{Det(s)}\delta\bigg(\frac{g-f}{2}\zeta+\frac{g+f}{2}\bigg)\frac{\varphi''L}{(1+\varphi'^2)^2}\Omega_\zeta$$

$$-\frac{1}{Det(s)}\delta\bigg(\frac{g'-f'}{2}\zeta-\frac{g'+f'}{2}\bigg)\frac{\varphi'L}{(1+\varphi'^2)}\Omega_\zeta, \tag{6.15}$$

which, by similar calculations completed for the previous change of coordinates contained in chapters 3 and 4 (and so omitted here), gives the following forms for ∇q and the velocity u:

$$\nabla q=\frac{\delta}{Det(s)}\frac{g(s)-f(s)}{2}\mathbf{t}(s)\Omega_s(s,\zeta,\tau)$$

$$+\frac{L}{Det(s)}\mathbf{n}(s)\Omega_\zeta(s,\zeta,\tau)$$

$$-\frac{\delta L}{Det(s)}\bigg(\frac{g(s)-f(s)}{2}\zeta+\frac{g(s)+f(s)}{2}\bigg)\frac{\varphi''(s)}{(1+\varphi'^2(s))^{3/2}}\mathbf{n}(s)\Omega_\zeta(s,\zeta,\tau)$$

$$-\frac{\delta L}{Det(s)}\bigg(\frac{g'(s)-f'(s)}{2}\zeta-\frac{g'(s)+f'(s)}{2}\bigg)\frac{1}{(1+\varphi'^2(s))^{1/2}}\mathbf{t}(s)\Omega_\zeta(s,\zeta,\tau) \tag{6.16}$$

$$\nabla^\perp q=\frac{\delta}{Det(s)}\frac{g(s)-f(s)}{2}\mathbf{n}(s)\Omega_s(s,\zeta,\tau)$$

$$-\frac{L}{Det(s)}\mathbf{t}(s)\Omega_\zeta(s,\zeta,\tau)$$

$$+\frac{\delta L}{Det(s)}\bigg(\frac{g(s)-f(s)}{2}\zeta+\frac{g(s)+f(s)}{2}\bigg)\frac{\varphi''(s)}{(1+\varphi'^2(s))^{3/2}}\mathbf{t}(s)\Omega_\zeta(s,\zeta,\tau)$$

$$-\frac{\delta L}{Det(s)}\bigg(\frac{g'(s)-f'(s)}{2}\zeta-\frac{g'(s)+f'(s)}{2}\bigg)\frac{1}{(1+\varphi'^2(s))^{1/2}}\mathbf{n}(s)\Omega_\zeta(s,\zeta,\tau) \tag{6.17}$$

where $\mathbf{n},\mathbf{t}$ and products of these terms are defined in Chapter 3 and the appendices. With the kernel $\tilde{K}_\alpha$ defined in Chapter 2 we have the velocity:

$$u(s,\zeta,\tau)=\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha(s,\bar{s},\zeta,\bar{\zeta})\nabla^\perp q(\bar{s},\bar{\zeta})Det(\bar{s})\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}$$

$$=\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha(s,\bar{s},\zeta,\bar{\zeta})\bigg[\delta\frac{g(\bar{s})-f(\bar{s})}{2}\mathbf{n}(\bar{s})\Omega_{\bar{s}}(\bar{s},\bar{\zeta},\tau)-L\mathbf{t}(\bar{s})\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)$$

$$+\delta L\left(\frac{g(\bar{s})-f(\bar{s})}{2}\bar{\zeta}+\frac{g(\bar{s})+f(\bar{s})}{2}\right)\frac{\varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}}\mathbf{t}(\bar{s})\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)$$
$$-\delta L\left(\frac{g'(\bar{s})-f'(\bar{s})}{2}\bar{\zeta}-\frac{g'(\bar{s})+f'(\bar{s})}{2}\right)\frac{1}{(1+\varphi'^2(\bar{s}))^{1/2}}\mathbf{n}(\bar{s})\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}. \quad (6.18)$$

6.2 Function Spaces

Our aim in this chapter is to show that, under the change of variables defined in (6.2), there is a time of existence independent of δ for which there are unique analytic solutions to the α-equation ($0 < \alpha < 1$) taking the form of an almost-sharp front. We will derive an IVP in this case and apply the version of the ACK theorem stated in Theorem 4.12, and as used in both Chapters 4 and 5.

In the previous existence results, on expansion of the term $u \cdot \nabla q$ we derived initial value problems for both cases, and, on defining appropriate function spaces (the strip and the bow-tie), we showed the required Cauchy estimates in order to satisfy the ACK theorem.

For the construction as above, with the two boundaries corresponding to $\zeta = \pm 1$, the natural function spaces to consider for $\zeta \in \mathbb{C}$ are functions that are analytic on the "double-ended pencil" (Figure 6.4). For $s \in \mathbb{C}$ we continue to study functions that are analytic on the strip.

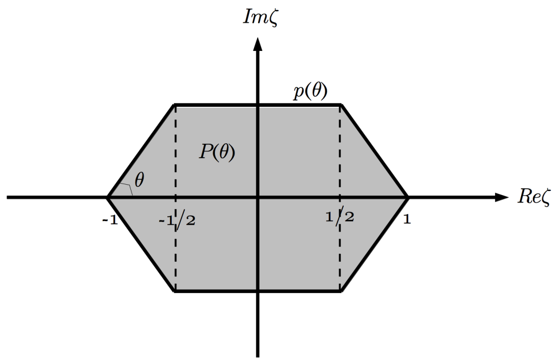

Figure 6.4: Double-Ended Pencil

In the study of problems with boundary, similar domains have been used, see for example Sammartino and Caflisch (1998b) in the study of the Navier-Stokes equation on the half-space. The function space $H^{l,\rho}$ for the strip is as given in Definition 4.15; we now introduce the L^2 spaces for functions analytic on the double ended pencil. Note that the same restrictions apply as in Sammartino and Caflisch (1998a) and Sammartino and Caflisch (1998b); l counts the number of derivatives, $l \geq 4$ and $0 < \theta \leq \frac{\pi}{4}$. We first define the domain:

$$\begin{aligned} P(\theta) &= \{\zeta \in \mathbb{C} : 0 \leq |\Re\zeta| \leq \frac{1}{2}, \quad 0 \leq |\Im\zeta| \leq \frac{1}{2}\tan\theta\} \\ &\cup\{\zeta \in \mathbb{C} : \frac{1}{2} \leq |\Re s| \leq 1, \quad |\Im\zeta| \leq (1 - |\Re\zeta|)\tan\theta\}, \end{aligned} \tag{6.19}$$

and the L^2 integration is taken along the following:

$$\begin{aligned} p(\theta') &= \{\zeta \in \mathbb{C} : 0 \leq |\Re\zeta| \leq \frac{1}{2}, \quad 0 \leq |\Im\zeta| = \frac{1}{2}\tan\theta'\} \\ &\cup\{\zeta \in \mathbb{C} : \frac{1}{2} \leq |\Re s| \leq 1, \quad |\Im\zeta| = (1 - |\Re\zeta|)\tan\theta'\}. \end{aligned} \tag{6.20}$$

The corresponding function spaces are defined in a similar fashion to those of chapters 4 and 5:

Definition 6.3. *For $f(\zeta)$ analytic on $P(\theta)$ for $\theta > 0$, the norm $\|f\|_{2,\theta}$ is given by:*

$$\|f\|_{2,\theta} = \sup_{|\theta'| \leq \theta} \|f(\cdot + iy)\|_{L^2(p(\theta'))}. \tag{6.21}$$

Definition 6.4. *Given $l \in \mathbb{N}$ and $\theta > 0$, $M^{l,\theta}$ is the set of all complex functions $f(\zeta)$ such that:*

- *f is analytic in $P(\theta)$.*
- *$\partial_\zeta^\alpha f \in L^2(p(\theta'))$ for $|\theta'| \leq \theta$.*
- *The norm $\|f\|_{2,l,\theta}$ is finite, where:*

$$\|f\|_{2,l,\theta} = \sum_{\alpha=0}^{l} \sup_{|\theta'| \leq \theta} \|\partial_\zeta^\alpha f\|_{2,\theta}. \tag{6.22}$$

Definition 6.5. *Given $l \in \mathbb{N}, \rho > 0, 0 < \theta \leq \frac{\pi}{4}$, $M^{l,\rho,\theta}$ is the set of all complex functions $f(s,\zeta)$ such that:*

- *f is analytic in $D(\rho) \times P(\theta)$ and periodic in $\Re s$.*
- *$\partial_\zeta^{\alpha_1}\partial_s^{\alpha_2} f \in L^2(p(\theta'); H^{0,\rho})$ for $|\theta'| \leq \theta$, $\alpha_1 + \alpha_2 \leq l$.*

- *The norm $\|f\|_{2,l,\rho,\theta}$ is finite, where:*

$$\|f\|_{2,l,\rho,\theta} = \sum_{\alpha_1+\alpha_2 \le l} \sup_{|\theta'| \le \theta} \left\| \|\partial_\zeta^{\alpha_1} \partial_s^{\alpha_2} f\|_{0,\rho} \right\|_{L^2(p(\theta'))}. \tag{6.23}$$

Definition 6.6. *Given $l \in \mathbb{N}$, $\rho, \beta, T > 0$ and $0 < \theta \le \frac{\pi}{4}$, the function $f(s,\zeta,t)$ is in $M_{\beta,T}^{l,\rho,\theta}$ if and only if*

- *$f(s,\zeta,t)$ is periodic in $\Re s$ and analytic in $D(\rho) \times P(\theta)$.*
- *$\partial_t^k f \in C([0,T]; M^{l-k,\rho,\theta})$ for $0 \le k \le l$.*
- *The norm $\|f\|_{2,l,\rho,\theta,\beta,T}$ is finite, where:*

$$\|f\|_{2,l,\rho,\theta,\beta,T} = \sum_{k=0}^{l} \sup_{0 \le t \le T} \|\partial_t^k f(\cdot,\cdot,t)\|_{2,l-k,\rho-\beta t,\theta-\beta t}. \tag{6.24}$$

where we have introduced the subsript 2 to distinguish between these norms and previous norms with the same number of parameters.

We hope to construct a solution that is constant outside of $\zeta \in [-1,1]$, and so this cannot be analytic in a small ball around the points $\zeta = \pm 1$. Hence as $\zeta \to \pm 1$, we can assume that the width of the strip of analyticity decreases to zero, and so Cauchy estimates considered in the same vein as in the previous chapters, for the above function spaces, do not hold in all of $P(\theta)$. Note that away from the small region around $\zeta = \pm 1$, the required Cauchy estimates can be shown by previous arguments.

We introduce the following result (analogous to one shown in Sammartino and Caflisch (1998a)):

Lemma 6.7. *Let $f, g \in M^{l,\theta}$ with $l \ge 4$ and $g(\zeta = 1) = g(\zeta = -1) = 0$, and let $\theta' < \theta$. Then:*

$$\|g\partial_\zeta f\|_{2,\theta'} \le \|g\|_{2,\theta} \frac{\|f\|_{2,\theta}}{\theta - \theta'}. \tag{6.25}$$

In order to utilise this result we split the region of analyticity for ζ into two parts, $0 \le \zeta < 1$ and $-1 < \zeta < 0$. We introduce the notation for the norms over the two regions; the L^2 integration remains the same, however is now taken over one of the intervals in ζ. Define:

$$p_+(\theta') = \{\zeta \in \mathbb{C}, 0 \leq \Re\zeta < 1 : 0 \leq \Re\zeta \leq \frac{1}{2}, \quad 0 \leq |\Im\zeta| = \frac{1}{2}\tan\theta'\}$$
$$\cup\{\zeta \in \mathbb{C}, 0 \leq \Re\zeta < 1 : \frac{1}{2} \leq \Re s \leq 1, \quad |\Im\zeta| = (1 - \Re\zeta)\tan\theta'\}, \tag{6.26}$$
$$p_-(\theta') = \{\zeta \in \mathbb{C}, -1 \leq \Re\zeta < 0 : 0 \leq |\Re\zeta| \leq \frac{1}{2}, \quad 0 \leq |\Im\zeta| = \frac{1}{2}\tan\theta'\}$$
$$\cup\{\zeta \in \mathbb{C}, -1 \leq \Re\zeta < 0 : \frac{1}{2} \leq \Re s \leq 1, \quad |\Im\zeta| = (1 - |\Re\zeta|)\tan\theta'\}, \tag{6.27}$$

and the corresponding norms for a function f analytic on $P(\theta)$:

$$\|f\|_{+,2,\theta} = \sup_{|\theta'|\leq\theta} \|f(\cdot + iy)\|_{L^2(p_+(\theta'))}, \tag{6.28}$$
$$\|f\|_{-,2,\theta} = \sup_{|\theta'|\leq\theta} \|f(\cdot + iy)\|_{L^2(p_-(\theta'))}, \tag{6.29}$$

with the function spaces and other norms adapted in each domain similarly. Using the construction and function spaces outlined in this section and §6.1 we now aim to rewrite the α-equation in the new variables in each of the two regions, and derive an IVP for which we show an existence result.

6.3 Initial Value Problem

In this section we rewrite the α-equation in the new coordinates, for the region $0 \leq \zeta < 1$ (indicated by $+$) as previously described. We construct an IVP for this region and give an existence result. The derivation for the region $-1 < \zeta < 0$ is analogous. Recall that for the $+$ region, $\zeta = 1$ corresponds to the top curve g. We aim to rewrite the equation in such a way that we can apply Lemma 6.7.

Having derived the forms of u and ∇q in (6.18) and (6.16) respectively, without writing these in full we have the equation for Ω in the new variables:

$$\Omega_t + \frac{I}{Det(s)}\Omega_s + \frac{II}{Det(s)}\Omega_\zeta + u(s,\zeta,t)\cdot\nabla q = 0 \tag{6.30}$$

which we rewite as:

$$\Omega_t + \frac{I}{Det(s)}\Omega_s + \frac{II}{Det(s)}\Omega_\zeta + u(s,1,t)\cdot\nabla q$$
$$+[u(s,\zeta,t) - u(s,1,t)]\cdot\nabla q = 0. \tag{6.31}$$

By the same arguments in Chapters 4 and 5, the Cauchy estimates required can be obtained for the term $\frac{I}{Det(s)}$; by the structure of I there are no terms of the form $\zeta\Omega_s$ and no derivatives in ζ. The Cauchy estimates on the final term $[u(s,\zeta,t)-u(s,1,t)]\cdot\nabla q$ are also possible on application of Lemma 6.7 as $[u(s,\zeta,t)-u(s,1,t)]$ vanishes at $\zeta=1$. It remains to study the Cauchy estimates on the terms that arise from $\frac{II}{Det(s)}\Omega_\zeta+u(s,1,t)\cdot\nabla q$. Utilising the level sets f and g we show that there is some cancellation in these terms to give the required estimates. We now derive the equations for f and g.

6.3.1 Equations for f and g

When $\zeta=1$ we have $\xi=g$; given g is defined to be a level set, it is advected with the fluid and satisfies:

$$\partial_t\left[(R^{-1}(s,t),\varphi(R^{-1}(s,t),t))+\delta g(s)\frac{(-\varphi',1)}{(1+\varphi'^2(s))^{1/2}}\right]=u(s,1,t), \tag{6.32}$$

see, for example, Majda and Bertozzi (2002). This gives:

$$\begin{aligned}(R_t^{-1},\varphi' R_t^{-1}+\varphi_t)+\delta[g'R_t^{-1}+g_t]\frac{(-\varphi',1)}{(1+\varphi'^2(s))^{1/2}}&\\ +\delta g\left(-\frac{\varphi'' R_t^{-1}+\varphi_t'}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'^2[\varphi'' R_t^{-1}+\varphi_t']}{(1+\varphi'^2(s))^{3/2}},-\frac{\varphi'[\varphi'' R_t^{-1}+\varphi_t']}{(1+\varphi'^2(s))^{3/2}}\right)&\\ =u(s,1,t).&\end{aligned} \tag{6.33}$$

We note that:

$$-\frac{\varphi'' R_t^{-1}+\varphi_t'}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'^2[\varphi'' R_t^{-1}+\varphi_t']}{(1+\varphi'^2(s))^{3/2}}=-\frac{\varphi'' R_t^{-1}+\varphi_t'}{(1+\varphi'^2(s))^{3/2}}$$

and rewrite (6.33) as:

$$R_t^{-1}\mathbf{t}(s)+(0,1)\varphi_t+\delta[g'R_t^{-1}+g_t]\mathbf{n}(s)-\delta g\frac{\varphi'' R_t^{-1}+\varphi_t'}{(1+\varphi'^2(s))}\mathbf{t}(s)=u(s,1,t), \tag{6.34}$$

with $\mathbf{n}$ and $\mathbf{t}$ as previously defined. On taking the dot product of (6.34) with $\mathbf{n}(s)$ we obtain:

$$\frac{\varphi_t}{(1+\varphi'^2(s))^{1/2}} + \delta[g' R_t^{-1} + g_t] = u(s,1,t)\cdot \mathbf{n}(s) \tag{6.35}$$

and so:

$$g_t = \frac{1}{\delta}\left[u(s,1,t)\cdot \mathbf{n}(s) - \frac{\varphi_t}{(1+\varphi'^2(s))^{1/2}}\right] - g' R_t^{-1}. \tag{6.36}$$

Using the expression for $u(s,\zeta,t)$ in (6.18), setting $\zeta = 1$ we obtain:

$$\begin{gathered}
u(s,1,t)\cdot \mathbf{n}(s) = \\
\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} \tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})\Bigg[\delta\frac{g(\bar{s})-f(\bar{s})}{2}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{s}}(\bar{s},\bar{\zeta},\tau) \\
+L\frac{\varphi'(s)-\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}(\bar{s})\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau) \\
+\delta L\left(\frac{g(\bar{s})-f(\bar{s})}{2}\bar{\zeta}+\frac{g(\bar{s})+f(\bar{s})}{2}\right) \\
\times\frac{\varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}}\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau) \\
-\delta L\left(\frac{g'(\bar{s})-f'(\bar{s})}{2}\bar{\zeta}-\frac{g'(\bar{s})+f'(\bar{s})}{2}\right) \\
\times\frac{1}{(1+\varphi'^2(\bar{s}))^{1/2}}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi},
\end{gathered} \tag{6.37}$$

$$\begin{gathered}
g_t = -g' R_t^{-1} + \frac{1}{\delta}\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} \tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})L\frac{\varphi'(s)-\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}(\bar{s})\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \\
-\frac{1}{\delta}\frac{\varphi_t}{(1+\varphi'^2(s))^{1/2}} \\
+\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} \tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})\frac{g(\bar{s})-f(\bar{s})}{2}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{s}}(\bar{s},\bar{\zeta},\tau)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \\
+\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} \tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})L\Bigg[\left(\frac{g(\bar{s})-f(\bar{s})}{2}\bar{\zeta}+\frac{g(\bar{s})+f(\bar{s})}{2}\right) \\
\times\frac{\varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}}\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}
\end{gathered}$$

$$
\begin{aligned}
&-\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} \tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})\Big[L\Big(\frac{g'(\bar{s})-f'(\bar{s})}{2}\bar{\zeta}-\frac{g'(\bar{s})+f'(\bar{s})}{2}\Big)\\
&\times\frac{1}{(1+\varphi'^2(\bar{s}))^{1/2}}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)\Big]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}. \qquad (6.38)
\end{aligned}
$$

Combining the second and third terms by use of the sharp front equation (2.11) we have:

$$
\begin{aligned}
g_t &= -g'R_t^{-1}\\
&+\frac{1}{\delta}\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}}[\tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})-\tilde{K}_\alpha(\delta=0)]L\frac{\varphi'(s)-\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}(\bar{s})\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}\\
&+\iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}}\tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})\Big[\frac{g(\bar{s})-f(\bar{s})}{2}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{s}}(\bar{s},\bar{\zeta},\tau)\\
&+L\Big[\Big(\frac{g(\bar{s})-f(\bar{s})}{2}\bar{\zeta}+\frac{g(\bar{s})+f(\bar{s})}{2}\Big)\\
&\times\frac{\varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}}\frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)\Big]\\
&-\Big[L\Big(\frac{g'(\bar{s})-f'(\bar{s})}{2}\bar{\zeta}-\frac{g'(\bar{s})+f'(\bar{s})}{2}\Big)\\
&\times\frac{1}{(1+\varphi'^2(\bar{s}))^{1/2}}\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2)^{1/2}}\Omega_{\bar{\zeta}}(\bar{s},\bar{\zeta},\tau)\Big]\Big]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}. \qquad (6.39)
\end{aligned}
$$

The equation for f is the same with 1 replaced by -1.

Remark 6.8. *Note that for $\alpha = 1$ some of the terms in the equations for f and g will contain form $\zeta\log\zeta$ for which the Cauchy estimates will not hold.*

6.3.2 Equation for Ω

In deriving the equation for Ω_t we need to make sense of the two terms $u(s,1,t)\cdot\nabla q$ that appear in (6.30) and so we must study the forms of $u(s,1,t)\cdot\mathbf{n}(s)$ and $u(s,1,t)\cdot\mathbf{t}(s)$. The first of these is given in (6.35):

$$
u(s,1,t)\cdot\mathbf{n}(s)=\frac{\varphi_t}{(1+\varphi'^2(s))^{1/2}}+\delta[g'R_t^{-1}+g_t]. \qquad (6.40)
$$

The latter is found by taking the dot product of (6.34) and $\mathbf{t}(s)$:

$$u(s,1,t)\cdot\mathbf{t}(s) = R_t^{-1}(1+\varphi'^2(s))^{\frac{1}{2}} + \frac{\varphi'\varphi_t}{(1+\varphi'^2(s))^{1/2}} - \delta g\frac{\varphi'' R_t^{-1}+\varphi_t'}{(1+\varphi'^2(s))}. \tag{6.41}$$

Using these terms we find that:

$$\begin{aligned}
&u(s,1,t)\cdot\nabla q = \\
&\frac{1}{Det(s)}\Bigg[\delta u(s,1,t)\cdot\mathbf{t}(s)\frac{g(s)-f(s)}{2}\Omega_s \\
&+Lu(s,1,t)\cdot\mathbf{n}\Omega_\zeta \\
&-\delta g\frac{\varphi''}{(1+\varphi'^2(s))^{3/2}}Lu(s,1)\cdot\mathbf{n}\Omega_\zeta \\
&-\delta g'\frac{L}{(1+\varphi'^2(s))^{1/2}}u(s,1)\cdot\mathbf{t}\Omega_\zeta\Bigg] \\
&= \frac{1}{Det(s)}\Omega_\zeta\Bigg[\frac{L\varphi_t}{(1+\varphi'^2(s))^{1/2}} + L\delta g_t - \frac{\delta Lg\varphi''\varphi_t}{(1+\varphi'^2(s))^2} \\
&-\delta Lg'\frac{\varphi'\varphi_t}{(1+\varphi'^2(s))} \\
&-\delta^2 Lg\frac{\varphi''}{(1+\varphi'^2(s))^{3/2}}(g'R_t^{-1}+g_t) + \delta^2 Lgg'\frac{1}{(1+\varphi'^2(s))^{3/2}}(\varphi'' R_t^{-1}+\varphi_t)\Bigg] \\
&+\frac{1}{Det(s)}\delta u(s,1,t)\cdot\mathbf{t}(s)\frac{g(s)-f(s)}{2}\Omega_s
\end{aligned} \tag{6.42}$$

and:

$$\begin{aligned}
&\frac{II(s,1)}{Det(s,\zeta)}\Omega_\zeta = \\
&\frac{\Omega_\zeta}{Det(s,\zeta)}\Bigg[-\frac{\varphi' L}{(1+\varphi'^2(s))^{1/2}} + \frac{\delta g\varphi_t\varphi'' L}{(1+\varphi'^2(s))^2} \\
&+R_t^{-1}\delta g' L + \varphi_t\delta g'\frac{\varphi' L}{(1+\varphi'^2(s))} \\
&-\delta(g'R_t^{-1}+g_t)L + \delta^2 g(g'R_t^{-1}+g_t)\frac{\varphi'' L}{(1+\varphi'^2(s))^{3/2}} \\
&-\delta^2 Lgg'\frac{\varphi'' R_t^{-1}+\varphi_t'}{(1+\varphi'^2(s))^{3/2}}\Bigg],
\end{aligned} \tag{6.43}$$

for which we note that the terms in (6.43) are the same as for the coefficients of Ω_ζ in (6.42) with opposite sign. This gives precisely:

$$\frac{II(s,1)}{Det(s,\zeta)}\Omega_\zeta + u(s,1)\cdot\nabla q = \frac{1}{Det(s,\zeta)}\delta u(s,1,t)\cdot\mathbf{t}(s)\frac{g(s)-f(s)}{2}\Omega_s$$
$$= \frac{1}{Det(s,\zeta)}\delta\frac{g-f}{2}\left[R_t^{-1}(1+\varphi'^2(s))^{\frac{1}{2}} + \frac{\varphi'\varphi_t}{(1+\varphi'^2(s))^{1/2}} - \delta g\frac{\varphi'' R_t^{-1}+\varphi_t'}{(1+\varphi'^2(s))}\right]$$
$$= -\frac{1}{Det(s,\zeta)}I(s,1). \tag{6.44}$$

We have now introduced the notation $Det(s,\zeta)$ to highlight its dependence on ζ. We rewrite the equation (6.31) for Ω as follows:

$$\Omega_t + \frac{I}{Det(s,\zeta)}\Omega_s + \frac{II}{Det(s,\zeta)}\Omega_\zeta + u\cdot\nabla q$$
$$= \Omega_t + \frac{I(s,\zeta)-I(s,1)}{Det(s)}\Omega_s + \frac{II(s,\zeta)-II(s,1)}{Det(s,\zeta)}\Omega_\zeta$$
$$+[u(s,\zeta,t)-u(s,1,t)]\cdot\nabla q \tag{6.45}$$

substituting in (6.44).

6.3.3 Cauchy Estimates

It remains to check the Cauchy estimates of the terms in (6.45). Whenever there is a product of terms we refer to the lemmas presented in Chapter 4 and Chapter 5 for the function spaces considered in those cases (e.g. Lemmas 4.20 - 4.26). The Cauchy estimates for the new spaces $M^{l,\rho,\theta}$ presented here follow from the analogous lemmas which we do not state.

For the terms in (6.45), the Cauchy estimates for the second and third terms hold; for the region of analyticity in s (the strip) the estimates for Ω_s are automatic and for the third term the estimates follow from Lemma 6.7.

It remains to prove in detail the estimates on the final term, that is:

$$[u(s,\zeta,t)-u(s,1,t)]\cdot\nabla q. \tag{6.46}$$

By definition of the velocity, this is precisely:

$$\iint_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} [\tilde{K}_\alpha(s,\bar{s},\zeta,\bar{\zeta}) - \tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})]\nabla^\perp q\,\mathrm{d}\bar{s}\mathrm{d}\bar{\zeta}\cdot\nabla q \tag{6.47}$$

$$= \iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} [\tilde{K}_\alpha(s,\bar{s},\zeta,\bar{\zeta}) - \tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta}) - \tilde{K}_\alpha(s,\bar{s},\zeta,1) + \tilde{K}_\alpha(s,\bar{s},1,1)]\nabla^\perp q \mathrm{d}\bar{s}\mathrm{d}\bar{\zeta} \cdot \nabla q$$

$$+ \iint\limits_{\mathbb{R}/\pi\mathbb{Z}\times\mathbb{R}} [\tilde{K}_\alpha(s,\bar{s},\zeta,1) - \tilde{K}_\alpha(s,\bar{s},1,1)]\nabla^\perp q \mathrm{d}\bar{s}\mathrm{d}\bar{\zeta} \cdot \nabla q, \tag{6.48}$$

where $\nabla^\perp q$ is given in (6.17). The term $[\tilde{K}_\alpha(s,\bar{s},\zeta,1)-\tilde{K}_\alpha(s,\bar{s},1,1)]$ is independent of $\bar{\zeta}$ and $\int_\mathbb{R} \nabla^\perp q \mathrm{d}\bar{\zeta}$ contains no derivatives of Ω with respect to $\bar{\zeta}$ by the properties:

$$\int \Omega_{\bar{\zeta}} \mathrm{d}\bar{\zeta} = 1, \quad \int \bar{\zeta}\Omega_{\bar{\zeta}} \mathrm{d}\bar{\zeta} = -\int \Omega \mathrm{d}\bar{\zeta}, \tag{6.49}$$

and so we can obtain the Cauchy estimates on all terms that arise. In (6.47) we have rewritten the integrand to ensure that $[\tilde{K}_\alpha(s,\bar{s},\zeta,\bar{\zeta}) - \tilde{K}_\alpha(s,\bar{s},1,\bar{\zeta})]$ vanishes at $\zeta = 1$ and that the integrand vanishes at $\bar{\zeta} = 1$; the Cauchy estimates then follow for all terms containing $\bar{\xi}$ by Lemma 6.7. The Cauchy estimates for terms of the form $\bar{\zeta}\Omega_{\bar{\zeta}}$ hold as presented in Chapter 5.

6.4 Existence Result

Writing the IVP as outlined in the previous sections we have:

$$\begin{cases} \Omega_t = H(t,\Omega,f,g), \quad f_t = F(t,\Omega,f,g), \quad g_t = G(t,\Omega,f,g) \\ \qquad \Omega|_{t=0} = \Omega_0, \quad f|_{t=0} = f_0, \quad g|_{t=0} = g_0 \end{cases} \tag{6.50}$$

where F, G and H have been constructed in (6.39) and (6.45). A similar construction holds for the IVP in the domain $-1 < \zeta < 0$; we omit the details.

Using the same methods as in the previous chapter in setting up the IVP, for which we show existence and uniqueness, we can introduce functions U, V and W that are contained in a related Banach space and set $H = \Omega_0 + W, F = f_0 + U, G = g_0 + V$ ensuring that all initial conditions are satisfied and that we are able to apply the ACK theorem (Theorem 4.12).

The difference between the existence proof presented here and those contained in Chapters 4 and 5 has been the Cauchy estimates in the corner of our domain. On constructing the new function spaces (see §6.2) and showing the estimates required in addition to those shown in Chapters 4 and 5, we have proved the following:

Theorem 6.9 (Existence of Analytic Almost-Sharp Front Solutions)**.** *Let* $\Omega_0, f_0, g_0 \in M^{l,\rho,\theta}$*,* $l \geq 4$ *and* $0 < \theta \leq \frac{\pi}{4}$*. Then (6.50) has a unique solution* $\Omega, f, g \in M^{l,\rho_0,\theta_0}_{\beta_0,T}$

for some $0 < \rho_0 < \rho, 0 < \theta_0 < \theta, \beta_0 > 0$ *and* $T > 0$. *In particular,* T *is independent of* δ.

By similar arguments to those of Chapters 4 and 5, and by the construction of F, G and H we are able to show that all terms are uniformly bounded in δ if we restrict to the case of small δ, $0 < \delta < \delta_0$, then we again have obtained solutions for a time interval independent of the size of the front. $\square$

Chapter 7

Conclusion

As outlined in the introduction, the interest in the open problem for the existence of singularities in finite time for solutions to the Euler and Navier Stokes equations has led to the study of sharp-fronts to the SQG equations, which is a two-dimensional system that, although simpler, retains many of the features of 3D Euler. The study of almost-sharp fronts for the SQG equation, and the study in the limit as the thickness δ approaches 0, has been introduced in order to study vortex lines for the SQG equation utilising methods that are not available in the 3D Euler (Córdoba et al., 2004).

This thesis has continued previous work on the construction of these almost-sharp fronts. The α-equation, when $0 < \alpha < 1$, has been defined as an interpolation model between the 2D Euler equation and SQG. The study of almost-sharp fronts for this equation has been continued beyond what is known currently for the SQG equations due to the less singular nature of kernel of the velocity in this case. This has provided the basis for much of the work presented. The interest in studying this system is primarily is to see which of the results can be extended to the case when $\alpha = 1$. We now provide a brief summary of the results presented and a series of open problems for further study.

For almost-sharp fronts defined as a regularisation of a sharp front in some δ-neighbourhood of a given curve φ, in Chapter 2 we derived an evolution equation for such a curve; this satisfied the sharp front equation given in (2.10) and (2.11) up to some error of order δ. We then showed that for an intrinsically defined curve the 'spine' the associated evolution also satified the sharp front equation up to an error of order δ^2. The errors derived when $\alpha < 1$ were much better than those previously shown for the SQG case; $\delta \log \delta$ and $\delta^2 \log \delta$ respectively. With coefficients of the sharp-front equation depending on α, the limiting nature as $\alpha \to 1$ of these evolution equations could be studied. One such open question could be to see which of the estimates for the SQG equation can be recovered in this limit.

The results contained in Chapter 2 all assumed the existence of almost sharp-front solutions and so of most interest was the study of their construction. For the SQG equation, the existence of almost-sharp front solutions remains an open problem; the most ideal result would be to show that there exist smooth almost-sharp front solutions to (2.1)-(2.2) and study these in the limit as $\alpha \to 1$. Unfortunately this also remains open; however for the α-equation we have seen that under analytic assumptions there do exist analytic solutions, both approximate and exact to almost-sharp fronts (Chapters 4 - 6).

For the smooth case, the construction of a family of almost-sharp fronts to the α-equation was studied in Chapter 3 as an analogue of Fefferman and Rodrigo (2012). We were unable to show existence of smooth solutions to the limiting equation presented here. One approach therefore would be to continue to study the well-posedness of the IVP associated to this limit equation; the hope being that if solutions could be found then we could use a limiting procedure and standard energy methods as $\alpha \to 1$ in order to study the SQG case. In Majda and Bertozzi (2002) and Kato (1984) this approach has been used when studying solutions to the Euler and Navier-Stokes equations as $\nu \to 0$. The existence of smooth solutions for the limit equation is however connected to the well-posedness of the Prandtl equations; for an overview of the current results known for this system see Grenier (2004).

The existence results presented in Chapters 4 - 6 ensure that for suitable initial conditions, utilising the ACK theorem, there exist both approximate solutions and exact solutions to the α-equation when posed on the two-dimensional cylinder, in the analytic case. The interest in studying such solutions is in the comparison to the case when $\alpha = 1$, and in introducing a method that may be extended to prove existence of the same type of solution for the SQG equation. The results in Chapters 4 and 5 cannot be extended directly at present to the SQG equations due to the presence of a logarithmic term, $\xi \log \xi \Omega_\xi$; the Cauchy estimates required to use the ACK theorem do not hold for such terms.

In the final chapter a new method is introduced that improves on the results in Chapter 4 and 5 and shows that there exist analytic solutions taking the form of an almost-sharp front; it remains to show that this method can be applied to the SQG equations to prove existence of analytic almost-sharp front solutions for that system.

Appendix A

Mathematical Techniques

This chapter contains an overview of some of the mathematical tools employed within the thesis. For the α-equation in (2.1) - (2.2), the velocity is defined as a convolution, $u = K * q$ for some kernel defined in §2.1. Properties of the convolution of two functions are discussed in §A.1, including Young's inequality for convolution as used frequently in the main text.

The existence results presented in Chapters 4 - 6 utilise a version of the Abstract Cauchy-Kovalevskaya (ACK) theorem detailed in §4.3. For comparison, we introduce a version of the standard Cauchy-Kovalevskaya (CK) theorem in §A.2.

A.1 Convolutions

The convolution of two function u and v on $\mathbb{R}^n$ is defined by:

$$u * v(x) = \int\limits_{\mathbb{R}^n} u(x-y)v(y)\mathrm{d}y = \int\limits_{\mathbb{R}^n} u(y)v(x-y)\mathrm{d}y$$

when the integral exists. An overview of properties satisfied by convolutions is given in Friedlander and Joshi (1998). In particular we use the following:

$$\partial_{x_j}(u * v) = \partial_{x_j} u * v = u * \partial_{x_j} v.$$

Recall $L^p(\Omega)$ is the space of all measurable functions u on Ω in $\mathbb{R}^n$ with standard norm:

$$\|u\|_p = \int\limits_{\Omega} |u(x)|^p \mathrm{d}x < \infty.$$

For functions belonging to such spaces, the following result holds; the proof can be found in Adams and Fournier (2003).

Theorem A.1 (Young's inequality for convolution)**.** *If* $(\frac{1}{p}) + (\frac{1}{q}) = 1 + (\frac{1}{r})$*, and if* $u \in L^p(\mathbb{R}^n)$ *and* $v \in L^q(\mathbb{R}^n)$*, then* $u * v \in L^r(\mathbb{R}^n)$*, and*

$$\|u * v\|_r \leq \|u\|_p \|v\|_q. \tag{A.1}$$

A.2 The Cauchy-Kovalevskaya Theorem

The CK theorem ensures existence and uniqueness of solutions to a system of partial differential equations with analytic Cauchy data; the statement of the theorem below is as presented in Folland (2005). Consider the Cauchy problem:

$$\begin{aligned} &\partial_t^k u(x,t) = G(x,t,(\partial_x^\alpha \partial_t^j u)_{|\alpha|+j \leq k, j<k}) \\ &\partial_t^j(x,0) = \phi_j(x) \quad (0 \leq j < k) \end{aligned} \tag{A.2}$$

where G, ϕ_j and u can be vector valued, then we have the following existence result:

Theorem A.2 (The Cauchy-Kovalevskaya Theorem)**.** *If* $G, \phi_0, \ldots, \phi_{k-1}$ *are analytic near the origin, there is a neighbourhood of the origin on which the Cauchy problem (A.2) has a unique analytic solution.*

We now present a version of the ACK theorem as outlined in Safonov (1995); this is the version adapted in Sammartino and Caflisch (1998a) and outlined in §4.3. Consider the problem:

$$u_t = F(u(t),t), \; u(0) = 0 \tag{A.3}$$

in a one parameter scale of Banach Spaces, $\{B_s, 0 < s < s_0\}$, such that $B_s \subset B_{s'}, \|\cdot\|_s' \leq \|\cdot\|_s$, for $0 < s' \leq s < s_0$, where $\|\cdot\|_s$ denotes the norm on B_s. Imposing the following assumptions on $F(u,t)$:

- For some constants $s_0 > 0, r > 0, \lambda > 0$, and every pair of numbers s, s' such that $0 < s' < s < s_0$, $0 \leq t \leq \frac{s_0}{\lambda}$, the correspondence $(u,t) \mapsto F(u,t)$ is a continuous mapping of

$$\{u \in B_s : \|u\|_s \leq r\} \times [0, \frac{s_0}{\lambda}) \text{ into } B_{s'}.$$

- For any $0 < s' < s < s_0, 0 \leq t < \frac{s_0}{\lambda}$, and for all $u, v \in B_s$ with $\|u\|_s < r, \|v\|_s < r$, we have:

$$\|F(u,t) - F(v,t)\|_{s'} \leq \frac{C}{s-s'}\|u-v\|_s, \tag{A.4}$$

where C is a constant independent of s, s', t, u, v.

- $F(0,t)$ is a continuous function of $t \in [0, \frac{s_0}{\lambda})$ with values in $B_s, 0 < s < s_0$, satisfying, with a fixed constant K,

$$\|F(0,t)\|_s \leq K.$$

we have the following:

Theorem A.3 (ACK, Safonov (1995))**.** *For any positive s_0, r, C and K, there is a positive constant λ_0 such that under the preceding assumptions with $\lambda > \lambda_0$, there exists a unique continuously differentiable function $u(t)$ with values in B_s, $0 < s < s_0$, $\|u\|_s < r$, which is defined for $0 \leq t < \frac{s_0 - s}{\lambda}$ and satisfies the problem (A.3).*

The assumption of analyticity, as required for the CK theorem, is included in (A.4); such estimates are the natural analogues of the Cauchy estimates that an analytic function satisfies, which is why analyticity is not explicitly referred to in the statement of the ACK theorem.

In contrast to the CK theorem, which is a basic existence theorem for analytic solutions to PDES, the ACK theorem is applicable to equations that contain non-local operators (Caflisch, 1990). For example, in Sammartino and Caflisch (1998a) and Sammartino and Caflisch (1998b) the ACK theorem is applied to the Prandtl equations which is dissipative, and for which the CK theorem cannot be applied.

Finally, we note that the version presented here is the most optimal for the existence proofs presented in Chapters 4-6 of the thesis. The version of the ACK theorem given in Theorem A.3 preserves the domain of existence, where the iterative proof presented in the earlier papers of Nirenberg (1972) and Nishida (1977) the domain of existence shrinks. For more details on this see Caflisch (1990).

Appendix B

Smooth Change of Coordinates

The derivation of the limit equations in the smooth case, the subject of Chapter 3, requires the use of a change of coordinates as introduced in Fefferman and Rodrigo (2012). An overview of this method is given in §3.1 alongside statements of the terms required in order to rewrite the α-equation in the new coordinates. This chapter contains the details of the calculations for obtaining such terms.

We consider the α-equation (3.1)-(3.2) posed on a two-dimensional cylindrical domain $(x, y) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, with $t \in [0, T]$, and we associate with this system a family of almost-sharp fronts, that is weak solutions of the α-equation, of the form:

$$q(x, y, t) = \begin{cases} \frac{1}{2} & \text{if } y \geq \varphi(x, t) + \delta \\ \text{bounded} & \text{if } |y - \varphi(x, t)| < \delta \\ -\frac{1}{2} & \text{if } y \leq \varphi(x, t) - \delta \end{cases} \tag{B.1}$$

where $\varphi(x, t)$ is a given smooth curve, periodic of period π in the x-variable, and $\delta > 0$. We focus our attention on the δ-neighbourhood of the curve as described in (B.1). The renormalized arc length for φ is given by:

$$R(x, t) = \frac{1}{L(t)} \int_{-\frac{\pi}{2}}^{x} (1 + \varphi'^2(\bar{x}, t))^{1/2} \mathrm{d}\bar{x}, \tag{B.2}$$

where $L(t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \varphi'^2(\bar{x}, t))^{1/2} \mathrm{d}\bar{x}$ is the length of the curve. We use the prime notation for differentiation with respect to the first variable. The function R is invertible as a function of x, and we construct a map from $(s, \xi) \in [0, 1) \times [-1, 1]$ to the δ-neighbourhood of the above curve as follows:

$$(x, y) = (R^{-1}(s, t), \varphi(R^{-1}(s, t), t)) + \mathbf{n}(R^{-1}(s, t))\xi\delta, \tag{B.3}$$

where $\mathbf{n}(R^{-1}(s, t))$ is the unit normal to the curve $y = \varphi(x, t)$ at the point $R^{-1}(s, t)$, given by:

$$\mathbf{n}(R^{-1}(s,t)) = \frac{(-\varphi'(R^{-1}(s,t),t),1)}{\|(-\varphi'(R^{-1}(s,t),t),1)\|} \tag{B.4}$$

and the corresponding unit tangent vector:

$$\mathbf{t}(R^{-1}(s,t)) = \frac{(1,\varphi'(R^{-1}(s,t),t))}{\|(1,\varphi'(R^{-1}(s,t),t)\|}. \tag{B.5}$$

The family of sharp fronts, as introduced in (A.1) is indexed by δ. We introduce a new time variable τ and so study a family of solutions to the α-equation, in the new coordinate system, of the form:

$$q(x,y,t) = \Omega(s,\xi,\tau) \tag{B.6}$$

where, using (A.3):

$$\Omega(s,\xi,\tau) = \begin{cases} \frac{1}{2} & \xi \geq 1 \\ \text{smooth} & |\xi| < 1 \\ -\frac{1}{2} & \xi \leq -1 \end{cases} \tag{B.7}$$

The remainder of this chapter contains the calculations required to rewrite the α-equation, as given in (3.1)-(3.2), in the new variables. The study of the new terms that arise, and their corresponding behaviour in the limit as $\delta \to 0$, forms the bulk of §3.3. We have:

$$(x,y,t) = (R^{-1}(s,\tau),\varphi(R^{-1}(s,\tau),\tau),\tau) + \frac{(-\varphi'(R^{-1}(s,\tau),\tau),1)}{\|(-\varphi'(R^{-1}(s,\tau),\tau),1)\|}\xi\delta, \tag{B.8}$$

and simplify the notation by setting $\varphi(s) = \varphi(R^{-1}(s,\tau))$. We supress the arguments for ease of notation when there is no ambiguity. Then:

$$\begin{cases} x = R^{-1}(s,\tau) + \dfrac{-\varphi'(s)\xi\delta}{(1+(\varphi'^2(s)))^{1/2}} \\ y = \varphi(s) + \dfrac{\xi\delta}{(1+(\varphi'^2(s)))^{1/2}} \\ t = \tau \end{cases} \tag{B.9}$$

In the new notation we also set:

$$\mathbf{n}(R^{-1}(s,\tau)) = (n_1,n_2) = \left(\frac{-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}}, \frac{1}{(1+\varphi'^2(s))^{1/2}}\right),$$

$$\mathbf{t}(R^{-1}(s,\tau)) = (t_1,t_2) = \left(\frac{1}{(1+\varphi'^2(s))^{1/2}}, \frac{\varphi'(s)}{(1+\varphi'^2(s))^{1/2}}\right).$$

Using the chain rule we have the following relations:

$$\begin{pmatrix} \partial_s \\ \partial_\xi \\ \partial_\tau \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & 0 \\ \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & 0 \\ \frac{\partial x}{\partial \tau} & \frac{\partial y}{\partial \tau} & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_t \end{pmatrix}$$

That is:

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_t \end{pmatrix} = \frac{1}{Det} \begin{pmatrix} \frac{\partial y}{\partial \xi} & -\frac{\partial y}{\partial s} & 0 \\ -\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial s} & 0 \\ I & II & Det \end{pmatrix} \begin{pmatrix} \partial_s \\ \partial_\xi \\ \partial_\tau \end{pmatrix}$$

where

$$Det(s) = \frac{\partial x}{\partial s}\frac{\partial y}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial s}, \quad I = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \xi}, \quad II = \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial \tau}.$$

The dependence of the determinant on the new variable s is highlighted for future analysis. Using (B.9):

$$\frac{\partial x}{\partial s} = R_s^{-1} + \left[\frac{-\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2(s)\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\right]\xi\delta,$$

$$\frac{\partial y}{\partial s} = \varphi'(s)R_s^{-1} - \frac{\varphi'(s)\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\xi\delta, \quad \frac{\partial x}{\partial \xi} = \frac{-\varphi'(s)\delta}{(1+\varphi'^2(s))^{1/2}}, \quad \frac{\partial y}{\partial \xi} = \frac{\delta}{(1+\varphi'^2(s))^{1/2}},$$

$$\frac{\partial x}{\partial \tau} = R_\tau^{-1} + \left[\frac{-\varphi''(s)R_\tau^{-1} - \varphi'_\tau(s)}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2(s)(\varphi''(s)R_\tau^{-1} + \varphi'_\tau(s))}{(1+\varphi'^2(s))^{3/2}}\right]\xi\delta,$$

$$\frac{\partial y}{\partial \tau} = \varphi'(s)R_\tau^{-1} + \varphi_\tau(s) - \frac{\varphi'(s)(\varphi''(s)R_\tau^{-1} + \varphi'_\tau(s))}{(1+\varphi'^2(s))^{3/2}}\xi\delta.$$

The determinant of the first matrix is then given by:

$$
\begin{aligned}
Det(s) &= \frac{\partial x}{\partial s}\frac{\partial y}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial s} \\
&= \left[R_s^{-1} + \left[\frac{-\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2(s)\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\right]\xi\delta\right]\frac{\delta}{(1+\varphi'^2(s))^{1/2}} \\
&\quad - \left[\varphi'(s)R_s^{-1} - \frac{\varphi'(s)\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\xi\delta\right]\frac{-\varphi'(s)\delta}{(1+\varphi'^2(s))^{1/2}} \\
&= \frac{R_s^{-1}(1+\varphi'^2(s))\delta}{(1+\varphi'^2(s))^{1/2}} + \frac{-\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))}\xi\delta^2.
\end{aligned}
$$

R_s^{-1} can be calculated using the inverse function theorem as follows:

$$
R_s^{-1}(s,\tau) = \frac{1}{R_x(x,t)} = \frac{1}{\frac{1}{L(t)}(1+\varphi'^2(x,t))^{1/2}} = \frac{L}{(1+\varphi'^2(s))^{1/2}}
$$

giving the simplified form:

$$
Det(s) = L\delta - L\frac{\varphi''(s)\xi\delta^2}{(1+\varphi'^2(s))^{3/2}}. \tag{B.10}
$$

In the derivation of the limit equation, terms containing $\frac{1}{Det(s)}$, and variants on this, are common. Using a series expansion we are able to determine the following estimates in δ for such terms. Using (A.10):

$$
Det(s) = L\delta\left[1 - \frac{\varphi''(s)\xi\delta}{(1+\varphi'^2(s))^{3/2}}\right]
$$

which in this form gives

$$
\frac{1}{Det(s)} = \frac{1}{L\delta} + \frac{1}{L}\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}} + O(\delta)
$$

and

$$
\frac{\delta}{Det(s)} = \frac{1}{L} + O(\delta).
$$

The calculations required to determine I and II follow:

$$
\begin{aligned}
I &= \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \xi} \\
&= \left[\varphi'(s)R_\tau^{-1} + \varphi_\tau(s) - \frac{\varphi'(s)(\varphi''(s)R_\tau^{-1} + \varphi'_\tau(s))}{(1+\varphi'^2(s))^{3/2}}\xi\delta\right]\frac{-\varphi'(s)\delta}{(1+\varphi'^2(s))^{1/2}}
\end{aligned}
$$

$$
\begin{aligned}
&-\left[R_\tau^{-1}+\left[\frac{-\varphi'' R_\tau^{-1}-\varphi'_\tau}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'^2(\varphi'' R_\tau^{-1}+\varphi'_\tau)}{(1+\varphi'^2(s))^{3/2}}\right]\xi\delta\right]\frac{\delta}{(1+\varphi'^2(s))^{1/2}}\\
&=\frac{\delta}{(1+\varphi'^2(s))^{1/2}}\left[-(1+\varphi'^2(s))R_\tau^{-1}-\varphi'(s)\varphi_\tau(s)+\frac{\varphi''(s)R_\tau^{-1}+\varphi'_\tau(s)}{(1+\varphi'^2(s))^{1/2}}\xi\delta\right],
\end{aligned}
$$

$$
\begin{aligned}
II &= \frac{\partial x}{\partial\tau}\frac{\partial y}{\partial s}-\frac{\partial x}{\partial s}\frac{\partial y}{\partial\tau}\\
&=\left[R_\tau^{-1}+\left[\frac{-\varphi''(s)R_\tau^{-1}-\varphi'_\tau(s)}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'^2(s)(\varphi''(s)R_\tau^{-1}+\varphi'_\tau(s))}{(1+\varphi'^2(s))^{3/2}}\right]\xi\delta\right]\\
&\quad\times\left[\varphi'(s)R_s^{-1}-\frac{\varphi'(s)\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\xi\delta\right]\\
&\quad-\left[R_s^{-1}+\left[\frac{-\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{1/2}}+\frac{\varphi'^2(s)\varphi''(s)R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\right]\xi\delta\right]\\
&\quad\times\left[\varphi'(s)R_\tau^{-1}+\varphi_\tau(s)-\frac{\varphi'(s)(\varphi''(s)R_\tau^{-1}+\varphi'_\tau(s))}{(1+\varphi'^2(s))^{3/2}}\xi\delta\right]\\
&=-R_s^{-1}\varphi' R_\tau^{-1}-R_s^{-1}\varphi_\tau+\varphi' R_s^{-1}R_\tau^{-1}\\
&\quad+\frac{\xi^2\delta^2}{(1+\varphi'^2(s))^2}\left[-\varphi'' R_s^{-1}\varphi'(\varphi'' R_\tau^{-1}+\varphi'_\tau)-\varphi'\varphi'' R_s^{-1}(-\varphi'' R_\tau^{-1}-\varphi'_\tau)\right]\\
&\quad+\frac{\xi^2\delta^2}{(1+\varphi'^2)^3}\left[\varphi'^2\varphi'' R_s^{-1}\varphi'(\varphi'' R_\tau^{-1}+\varphi'_\tau)-\varphi'\varphi'' R_s^{-1}\varphi'^2(\varphi'' R_\tau^{-1}+\varphi'_\tau)\right]\\
&\quad+\frac{\xi\delta}{(1+\varphi'^2(s))^{1/2}}\left[\varphi''\varphi' R_\tau^{-1}R_s^{-1}+\varphi''\varphi_\tau R_s^{-1}-\varphi'\varphi'' R_s^{-1}R_\tau^{-1}-\varphi' R_s^{-1}\varphi'_\tau\right]\\
&\quad+\frac{\xi\delta}{(1+\varphi'^2(s))^{3/2}}\Big[\varphi'(\varphi'' R_\tau^{-1}+\varphi'_\tau)R_s^{-1}-\varphi' R_\tau^{-1}\varphi'^2\varphi'' R_s^{-1}-\varphi_\tau\varphi'^2\varphi'' R_s^{-1}\\
&\qquad+\varphi' R_s^{-1}\varphi'^2(\varphi'' R_\tau^{-1}+\varphi'_\tau)-\varphi'\varphi'' R_s^{-1}R_\tau^{-1}\Big]\\
&=-R_s^{-1}\varphi_\tau(s)+\frac{\xi\delta}{(1+\varphi'^2(s))^{1/2}}\left[\varphi''(s)\varphi_\tau R_s^{-1}-\varphi'(s)R_s^{-1}\varphi'_\tau\right]\\
&\quad+\frac{\xi\delta}{(1+\varphi'^2(s))^{3/2}}\left[\varphi'(s)\varphi'_\tau(s)R_s^{-1}-\varphi_\tau(s)\varphi'^2(s)\varphi''(s)R_s^{-1}+\varphi'^3(s)R_s^{-1}\varphi'_\tau(s)\right]\\
&=-R_s^{-1}\varphi_\tau(s)+\frac{\xi\delta}{(1+\varphi'^2(s))^{3/2}}\left[\varphi''(s)\varphi_\tau(s)R_s^{-1}\right]\\
&=-\frac{L\varphi_\tau(s)}{(1+\varphi'^2(s))^{1/2}}+\frac{L\varphi''(s)\varphi_\tau(s)}{(1+\varphi'^2(s))^2}\xi\delta.
\end{aligned}
$$

Using the previous estimates obtained on terms of the from $\frac{1}{Det(s)}$ we are able to give the forms of the derivatives of q in the new coordinate system. We first obtain that:

$$\frac{I}{Det(s)} = \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}\left[-(1+\varphi'^2(s))R_\tau^{-1} - \varphi'(s)\varphi_\tau\right] + O(\delta),$$

$$\frac{II}{Det(s)} = -\frac{1}{\delta}\frac{\varphi'_\tau(s)}{(1+\varphi'^2(s))^{1/2}} + 2\frac{\varphi''(s)\varphi_\tau(s)\xi\delta}{(1+\varphi'^2(s))^{1/2}} + O(\delta),$$

enabling us to determine:

$$\begin{aligned}
\partial_t q &= \frac{I}{Det(s)}\Omega_s + \frac{II}{Det(s)}\Omega_\xi + \Omega_\tau \\
&= \Omega_\tau(s,\xi,\tau) + \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}\left[-(1+\varphi'^2(s))R_\tau^{-1} - \varphi'(s)\varphi_\tau(s)\right]\Omega_s(s,\xi,\tau) \\
&\quad -\frac{1}{\delta}\frac{\varphi_\tau(s)}{(1+\varphi'^2(s))^{1/2}}\Omega_\xi(s,\xi,\tau) + 2\frac{\varphi''(s)\varphi_\tau(s)\xi}{(1+\varphi'^2(s))^2}\Omega_\xi(s,\xi,\tau) + O(\delta), \qquad \text{(B.11)}
\end{aligned}$$

$$\begin{aligned}
\partial_x q &= \frac{1}{Det(s)}\frac{\partial y}{\partial \xi}\Omega_s - \frac{1}{Det(s)}\frac{\partial y}{\partial s}\Omega_\xi \\
&= \frac{1}{Det(s)}\frac{\delta}{(1+\varphi'^2(s))^{1/2}}\Omega_s(s,\xi,\tau) \\
&\quad + \frac{1}{Det(s)}\left(\frac{-\varphi'(s)L}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'(s)\varphi''(s)L\xi\delta}{(1+\varphi'^2(s))^2}\right)\Omega_\xi(s,\xi,\tau), \qquad \text{(B.12)}
\end{aligned}$$

$$\begin{aligned}
\partial_y q &= -\frac{1}{Det(s)}\frac{\partial x}{\partial \xi}\Omega_s + \frac{1}{Det(s)}\frac{\partial x}{\partial s}\Omega_\xi \\
&= \frac{\varphi'(s)\delta}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\Omega_s(s,\xi,\tau) + \frac{L}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\Omega_\xi(s,\xi,\tau) \\
&\quad + \frac{\xi\delta}{Det(s)}\left[\frac{-\varphi''(s)L}{(1+\varphi'^2(s))} + \frac{\varphi'^2(s)\varphi''(s)L}{(1+\varphi'^2(s))^2}\right]\Omega_\xi(s,\xi,\tau) \\
&= \frac{\varphi'(s)\delta}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\Omega_s(s,\xi,\tau) + \frac{L}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\Omega_\xi(s,\xi,\tau) \\
&\quad - \frac{\delta\xi}{Det(s)}\frac{\varphi'' L}{(1+\varphi'^2(s))^2}\Omega_\xi(s,\xi,\tau), \qquad \text{(B.13)}
\end{aligned}$$

Recalling the definitions of the unit tangent $\mathbf{t}$ and the unit normal $\mathbf{n}$ to the curve φ, notice that:

$$\partial_x q = t_1 \frac{\delta}{Det(s)}\Omega_s + n_1 \frac{L}{Det(s)}\Omega_\xi - n_1 \frac{\delta\xi}{Det(s)} \frac{\varphi''(s)L}{(1+\varphi'^2(s))^{3/2}}\Omega_\xi,$$
$$\partial_y q = t_2 \frac{\delta}{Det(s)}\Omega_s + n_2 \frac{L}{Det(s)}\Omega_\xi - n_2 \frac{\delta\xi}{Det(s)} \frac{\varphi''(s)L}{(1+\varphi'^2(s))^{3/2}}\Omega_\xi,$$

and so:

$$\nabla q = \mathbf{t}\frac{\delta}{Det(s)}\Omega_s + \mathbf{n}\frac{L}{Det(s)}\Omega_\xi - \mathbf{n}\frac{\xi\delta}{Det(s)} \frac{\varphi''(s)L}{(1+\varphi'^2(s))^{3/2}}\Omega_\xi, \tag{B.14}$$

which, with the relations $\mathbf{t}^\perp = \mathbf{n}$ and $\mathbf{n}^\perp = -\mathbf{t}$, also gives:

$$\nabla^\perp q = \mathbf{n}\frac{\delta}{Det(s)}\Omega_s - \mathbf{t}\frac{L}{Det(s)}\Omega_\xi + \mathbf{t}\frac{\xi\delta}{Det(s)} \frac{\varphi''(s)L}{(1+\varphi'^2(s))^{3/2}}\Omega_\xi. \tag{B.15}$$

Let K_α^δ be the kernel defined in Chapter 3. Then with u as given in (3.2), and under the change of coordinates introduced in (A.9) we have the expression

$$u(s,\xi,\tau) = \iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} K_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})\nabla^\perp_{\bar{s},\bar{\xi}}\Omega(\bar{s},\bar{\xi})\cdot\nabla_{s,\xi}\Omega(s,\xi)Det(\bar{s})\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{B.16}$$

for which we need the following properties:

$$\mathbf{t}(s)\cdot\mathbf{t}(\bar{s}) = \frac{(1,\varphi'(s))}{(1+\varphi'^2(s))^{1/2}}\cdot\frac{(1,\varphi'(\bar{s}))}{(1+\varphi'^2(\bar{s}))^{1/2}} = \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}, \tag{B.17}$$

$$\mathbf{n}(s)\cdot\mathbf{n}(\bar{s}) = \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} = \mathbf{t}(s)\cdot\mathbf{t}(\bar{s}), \tag{B.18}$$

$$\mathbf{t}(s)\cdot\mathbf{n}(\bar{s}) = \frac{\varphi'(s)-\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}, \tag{B.19}$$

$$\mathbf{n}(s)\cdot\mathbf{t}(\bar{s}) = \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{\frac{1}{2}}(1+\varphi'^2(\bar{s}))^{1/2}} = -\mathbf{t}(s)\cdot\mathbf{n}(\bar{s}). \tag{B.20}$$

The product term reduces to:

$$\nabla^{\perp}_{\bar{s},\bar{\xi}}\Omega(\bar{s},\bar{\xi})\cdot\nabla_{s,\xi}\Omega(s,\xi)Det(\bar{s}) =$$

$$\left(\mathbf{n}(\bar{s})\delta\Omega_{\bar{s}} - \mathbf{t}(\bar{s})L\Omega_{\bar{\xi}} + \mathbf{t}(\bar{s})\frac{\varphi''(\bar{s})L\delta\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega_{\bar{\xi}}\right)\cdot$$

$$\left(\mathbf{t}(s)\frac{\delta}{Det(s)}\Omega_s + \mathbf{n}(s)\frac{L}{Det(s)}\Omega_\xi - \mathbf{n}(s)\frac{\varphi''(s)L\delta\xi}{Det(s)(1+\varphi'^2(s))^{3/2}}\Omega_\xi\right)$$

$$= \mathbf{n}(\bar{s})\cdot\mathbf{t}(s)\frac{\delta^2}{Det(s)}\Omega_{\bar{s}}\Omega_s + \mathbf{n}(\bar{s})\cdot\mathbf{n}(s)\frac{\delta L}{Det(s)}\Omega_{\bar{s}}\Omega_\xi$$

$$-\mathbf{n}(\bar{s})\cdot\mathbf{n}(s)\frac{\delta^2 L}{Det(s)}\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}\Omega_{\bar{s}}\Omega_\xi - \mathbf{t}(\bar{s})\cdot\mathbf{t}(s)\frac{\delta L}{Det(s)}\Omega_{\bar{\xi}}\Omega_s$$

$$-\mathbf{t}(\bar{s})\cdot\mathbf{n}(s)\frac{L^2}{Det(s)}\Omega_{\bar{\xi}}\Omega_\xi + \mathbf{t}(\bar{s})\cdot\mathbf{n}(s)\frac{\delta L^2}{Det(s)}\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}\Omega_{\bar{\xi}}\Omega_\xi$$

$$+\mathbf{t}(\bar{s})\cdot\mathbf{t}(s)\frac{\delta^2 L}{Det(s)}\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega_{\bar{\xi}}\Omega_s$$

$$+\mathbf{t}(\bar{s})\cdot\mathbf{n}(s)\frac{\delta L^2}{Det(s)}\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega_{\bar{\xi}}\Omega_\xi$$

$$-\mathbf{t}(\bar{s})\cdot\mathbf{n}(s)\frac{\delta^2 L^2}{Det(s)}\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}\Omega_{\bar{\xi}}\Omega_\xi$$

$$= \frac{1}{Det(s)}\Bigg[-\frac{(\varphi'(\bar{s})-\varphi'(s))\delta^2}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_s\Omega_{\bar{s}} + \frac{(1+\varphi'(s)\varphi'(\bar{s}))\delta L}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{s}}\Omega_\xi$$

$$-\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\delta^2 L\xi\varphi''(s)}{(1+\varphi'^2(s))^{3/2}}\Omega_\xi\Omega\bar{s}$$

$$-\frac{(1+\varphi'(s)\varphi'(\bar{s}))L\delta}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\Omega_s - \frac{(\varphi'(\bar{s})-\varphi'(s))L^2}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\Omega_{\bar{\xi}}\Omega_\xi$$

$$+\frac{(\varphi'(\bar{s})-\varphi'(s))}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{L^2\delta\xi\varphi''(s)}{(1+\varphi^2(s))^{3/2}}\Omega_\xi\Omega_{\bar{\xi}}$$

$$+\frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{L\delta^2\bar{\xi}\varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega_s\Omega_{\bar{\xi}}$$

$$+\frac{(\varphi'(\bar{s})-\varphi'(s))}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{L^2\delta\bar{\xi}\varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega\xi\Omega_{\bar{\xi}}$$

$$-\frac{(\varphi'(\bar{s})-\varphi'(s))}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}L^2\delta^2\Omega_\xi\Omega_{\bar{\xi}}\Bigg],$$

which on grouping some of the terms gives us the final expression for $u\cdot\nabla q$ as required for the limit equation, where the double integrals are taken over the domain $(\bar{s},\bar{\xi})\in\mathbb{R}/\mathbb{Z}\times[-1,1]$:

$$u \cdot \nabla q =$$

$$-\frac{L^2}{Det(s)} \iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_\xi \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{B.21}$$

$$+\frac{L\delta}{Det(s)} \iint K_\alpha^\delta \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \left[\Omega_\xi \Omega_{\bar{s}} - \Omega_s \Omega_{\bar{\xi}}\right] \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{B.22}$$

$$+\frac{L^2\delta}{Det(s)} \iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \left(\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} + \frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}} \right) \Omega_\xi \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{B.23}$$

$$-\frac{\delta^2}{Det(s)} \iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \Omega_s \Omega_{\bar{s}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{B.24}$$

$$+\frac{L\delta^2}{Det(s)} \iint K_\alpha^\delta \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} \left(\frac{\bar{\xi}\Omega_s \Omega_{\bar{\xi}} \varphi''(\bar{s})}{(1+\varphi'^2(\bar{s}))^{3/2}} + \frac{\xi \Omega_\xi \Omega_{\bar{s}} \varphi''(s)}{(1+\varphi'^2(s))^{3/2}} \right) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{B.25}$$

$$-\frac{L^2\delta^2}{Det(s)} \iint K_\alpha^\delta \frac{\varphi'(\bar{s}) - \varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{\frac{1}{2}}} \frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}} \times \frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}} \Omega_\xi \Omega_{\bar{\xi}} \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}. \tag{B.26}$$

Appendix C

Analytic Change of Coordinates

Chapter 4 contains the derivation of the limit equations in the analytic case, which requires a change of coordinates introduced in correspondence with Jose Rodrigo, and contained in current work between himself and Charles Fefferman. This chapter contains the calculations required in order to rewrite the α-equation (3.1)-(3.2) in the new variables.

Throughout this chapter we use the same notation as employed in Appendix A unless explicitly stated, or where new notation is introduced for further simplification. Using the same form of the weak solution (B.1), and the renormalised arc length for the curve φ, (B.2), we introduce the following change of coordinates, noting that we now consider $\xi \in \mathbb{R}$:

$$(x,y) = (R^{-1}(s,t), \varphi(R^{-1}(s,t),t)) + \left(n_1(R^{-1}(s,t)) \frac{\delta\xi}{1+100(\delta\xi)^{100}}, n_2(R^{-1}(s,t))\delta\xi \right) \tag{C.1}$$

which on introducing a new time variable τ, and simplifying notation gives:

$$\left\{ \begin{array}{lcc} x & = & R^{-1}(s) - \frac{\varphi'(s),\tau)}{(1+(\varphi'(s)^2)^{1/2}} \frac{\delta\xi}{1+100(\delta\xi)^{100}} \\ y & = & \varphi(s) + \frac{\delta\xi}{(1+(\varphi'(s)^2)^{1/2}} \\ t & = & \tau \end{array} \right. \tag{C.2}$$

The same calculations as in the smooth case give that:

$$\partial_x = \frac{1}{Det(s)} \frac{\partial y}{\partial \xi} \partial_s - \frac{1}{Det(s)} \frac{\partial y}{\partial s} \partial_\xi, \quad \partial_y = \frac{1}{Det(s)} \frac{\partial x}{\partial s} \partial_\xi - \frac{1}{Det(s)} \frac{\partial x}{\partial \xi} \partial_s,$$

$$\partial_t = \frac{I}{Det(s)}\partial_s + \frac{II}{Det(s)}\partial_\xi + \partial_\tau,$$

where:

$$Det(s) = \frac{\partial x}{\partial s}\frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial \xi}, \quad I = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \tau} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \tau}, \quad II = \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial \tau}\frac{\partial x}{\partial s}.$$

We obtain the following, suppressing the arguments when it is clear:

$$\begin{aligned}
\frac{\partial x}{\partial s} &= R_s^{-1} + \left[\frac{-\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} - \varphi'\left(\frac{\partial}{\partial s}\frac{1}{(1+\varphi'^2(s))^{1/2}}\right)\right]\frac{\delta\xi}{1+100(\delta\xi)^{100}} \\
&= R_s^{-1} + \left[\frac{-\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} - \frac{\varphi'(-\frac{1}{2})2\varphi'' R_s^{-1}\varphi'}{(1+\varphi'^2(s))^{3/2}}\right]\frac{\delta\xi}{1+100(\delta\xi)^{100}} \\
&= R_s^{-1} + \left[\frac{-\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\right]\frac{\delta\xi}{1+100(\delta\xi)^{100}},
\end{aligned} \tag{C.3}$$

$$\frac{\partial y}{\partial s} = \varphi' R_s^{-1} - \frac{\varphi'\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\delta\xi, \tag{C.4}$$

$$\begin{aligned}
\frac{\partial x}{\partial \xi} &= \frac{\partial}{\partial \xi}\left(\frac{\delta\xi}{1+100(\delta\xi)^{100}}\right)\left(\frac{-\varphi'}{(1+\varphi'^2(s))^{1/2}}\right) \\
&= \left[\frac{\delta}{1+100(\delta\xi)^{100}} + \delta\xi\frac{(-10^4\delta^{100}\xi^{99})}{(1+100(\delta\xi)^{100})^2}\right]\frac{-\varphi'}{(1+\varphi'^2(s))^{1/2}} \\
&= -\frac{\varphi'}{(1+\varphi'^2(s))^{1/2}}\left[\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right],
\end{aligned} \tag{C.5}$$

$$\frac{\partial y}{\partial \xi} = \frac{\delta}{(1+\varphi'(s)^2)^{1/2}}, \tag{C.6}$$

$$\begin{aligned}
\frac{\partial x}{\partial \tau} &= R_\tau^{-1} + \frac{\delta\xi}{1+100(\delta\xi)^{100}}\frac{\partial}{\partial \tau}\left(-\frac{\varphi'(R_s^{-1}(s,\tau),\tau)}{(1+\varphi'^2(s))^{1/2}}\right) \\
&= R_\tau^{-1} + \frac{\delta\xi}{1+100(\delta\xi)^{100}}\left[\frac{-\varphi'' R_\tau^{-1} - \varphi'_\tau}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2(\varphi'' R_\tau^{-1} + \varphi'_\tau)}{(1+\varphi'^2(s))^{3/2}}\right],
\end{aligned} \tag{C.7}$$

$$\frac{\partial y}{\partial \tau} = \varphi' R_\tau^{-1} + \varphi_\tau + \delta\xi\left(\frac{(-\frac{1}{2})2(\varphi'' R_\tau^{-1} + \varphi'_\tau)\varphi'}{(1+\varphi'^2(s))^{3/2}}\right)$$

$$= \varphi' R_\tau^{-1} + \varphi_\tau - \frac{\varphi'(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^{3/2}}. \tag{C.8}$$

Recall that:

$$R_s^{-1} = \frac{L}{(1+\varphi'^2(s))^{1/2}}$$

and so we can calculate the determinant $Det(s)$ as follows:

$$Det(s) = \frac{\partial x}{\partial s}\frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial \xi}$$

$$= \left(R_s^{-1} + \left[\frac{-\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{\frac{3}{2}}}\right]\frac{\delta\xi}{1+100(\delta\xi)^{100}}\right)\frac{\delta}{(1+\varphi'^2(s))^{1/2}}$$

$$+\left(\varphi' R_s^{-1} - \frac{\varphi'\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\delta\xi\right)\frac{\varphi'}{(1+\varphi'^2(s))^{1/2}}\left[\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right]$$

$$= \frac{R_s^{-1}\delta}{(1+\varphi'^2(s))^{1/2}} + \frac{-\varphi'' R_s^{-1}\delta^2\xi}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}$$

$$+\frac{\varphi'^2 R_s^{-1}}{(1+\varphi'^2(s))^{1/2}}\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{\varphi'^2 R_s^{-1}}{(1+\varphi'^2(s))^{1/2}}\left(\frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)$$

$$-\frac{\varphi'^2\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\delta\xi\frac{1}{(1+\varphi'^2(s))^{1/2}}\frac{\delta}{1+100(\delta\xi)^{100}}$$

$$+\frac{\varphi'^2\varphi'' R_s^{-1}\delta\xi}{(1+\varphi'^2(s))^{3/2}}\frac{1}{(1+\varphi'^2(s))^{1/2}}\frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}$$

$$= \frac{R_s^{-1}\delta}{(1+\varphi'^2(s))^{1/2}} - \frac{\varphi'' R_s^{-1}\delta^2\xi}{(1+\varphi'(s)^2)^2(1+100(\delta\xi)^{100})}$$

$$+\frac{\varphi'^2 R_s^{-1}\delta}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})} - \frac{\varphi'^2 R_s^{-1}10^4(\delta\xi)^{100}\delta}{(1+\varphi'(s)^2)^{1/2}(1+100(\delta\xi)^{100})^2}$$

$$-\frac{\varphi'^2\varphi'' R_s^{-1}\delta^2\xi}{(1+\varphi'(s)^2)^2(1+100(\delta\xi)^{100})} + \frac{\varphi'^2\varphi'' R_s^{-1}10^4(\delta\xi)^{101}\delta}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})^2}$$

$$= \frac{R_s^{-1}\delta}{(1+\varphi'^2(s))^{1/2}}$$

$$+\frac{\varphi'^2 R_s^{-1}\delta(1+100(\delta\xi)^{100})}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})} - \frac{\varphi'^2 R_s^{-1}\delta 100(\delta\xi)^{100}}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})}$$

$$-\frac{\varphi'^2 R_s^{-1}10^4(\delta\xi)^{100}\delta}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})^2} + \frac{\varphi'^2\varphi'' R_s^{-1}10^4(\delta\xi)^{101}\delta}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})^2}$$

$$-\frac{\varphi'' R_s^{-1}\delta^2\xi(1+\varphi'^2(s))}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}$$

$$= \frac{R_s^{-1}\delta}{(1+\varphi'(s)^2)^{1/2}} + \frac{\varphi'^2 R_s^{-1}\delta}{(1+\varphi'(s)^2)^{1/2}} - \frac{\varphi'^2 R_s^{-1}\delta 100(\delta\xi)^{100}}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})}$$

$$
\begin{aligned}
&-\frac{\varphi'^2 R_s^{-1} 10^4 (\delta\xi)^{100}\delta}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})^2}+\frac{\varphi'^2\varphi'' R_s^{-1} 10^4(\delta\xi)^{101}\delta}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})^2}\\
&-\frac{\varphi'' R_s^{-1}\delta^2\xi}{(1+\varphi'^2(s))(1+100(\delta\xi)^{100})}\\
&= L\delta-\frac{\varphi'^2 L\delta 100(\delta\xi)^{100}}{(1+\varphi'^2(s))(1+100(\delta\xi)^{100})}-\frac{\varphi'^2 L 10^4(\delta\xi)^{100}\delta}{(1+\varphi'^2(s))(1+100(\delta\xi)^{100})^2}\\
&+\frac{\varphi'^2 L\varphi'' 10^4(\delta\xi)^{101}\delta}{(1+\varphi'^2(s))^{5/2}(1+100(\delta\xi)^{100})^2}-\frac{\varphi'' L\delta^2\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})}.
\end{aligned}
$$

That is:

$$
\begin{aligned}
Det(s) =& L\delta\Big[1-\frac{\varphi''\delta\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})}-\frac{\varphi'^2 100(\delta\xi)^{100}}{(1+\varphi'^2(s))(1+100(\delta\xi)^{100})}\\
&-\frac{\varphi'^2 10^4(\delta\xi)^{100}}{(1+\varphi'^2(s))(1+100(\delta\xi)^{100})^2}+\frac{\varphi'^2\varphi'' 10^4(\delta\xi)^{101}}{(1+\varphi'^2(s))^{5/2}(1+100(\delta\xi)^{100})^2}\Big].
\end{aligned}
\tag{C.9}
$$

We now calculate the remaining terms:

$$
\begin{aligned}
I &= \frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\tau}-\frac{\partial y}{\partial\xi}\frac{\partial x}{\partial\tau}\\
&= \frac{-\varphi'}{(1+\varphi'^2(s))^{1/2}}\left(\frac{\delta}{1+100(\delta\xi)^{100}}-\frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\\
&\quad\times\left((\varphi' R_\tau^{-1}+\varphi_\tau)-\frac{\varphi'(\varphi'' R_\tau^{-1}+\varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^{3/2}}\right)\\
&\quad-\frac{\delta}{(1+\varphi'^2(s))^{1/2}}\left(R_\tau^{-1}+\frac{-\varphi'' R_\tau^{-1}-\varphi'_\tau}{(1+\varphi'^2(s))^{3/2}}\frac{\delta\xi}{1+100(\delta\xi)^{100}}\right)\\
&= \frac{-\varphi'\delta(\varphi' R_\tau^{-1}+\varphi_\tau)}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})}+\frac{\varphi'\delta\varphi'(\varphi'' R_\tau^{-1}+\varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\\
&\quad+\frac{\varphi' 10^4(\delta\xi)^{100}\delta(\varphi' R_\tau^{-1}+\varphi_\tau)}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})^2}-\frac{\varphi' 10^4(\delta\xi)^{100}\delta\varphi'(\varphi'' R_\tau^{-1}+\varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})^2}\\
&\quad-\frac{\delta R_\tau^{-1}}{(1+\varphi'^2(s))^{1/2}}+\frac{\delta(\varphi'' R_\tau^{-1}+\varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\\
&= \left(\frac{-\varphi'^2\delta R_\tau^{-1}-\varphi'\varphi_\tau\delta}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})}-\frac{\delta R_\tau^{-1}}{(1+\varphi'^2(s))^{1/2}}\right)\\
&\quad+\frac{(\varphi'' R_\tau^{-1}+\varphi'_\tau)\delta^2\xi}{(1+\varphi'^2(s))(1+100(\delta\xi)^{100})}+\frac{\varphi' 10^4(\delta\xi)^{100}\delta(\varphi' R_\tau^{-1}+\varphi_\tau)}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})^2}\\
&\quad-\frac{\varphi'^2 10^4(\delta\xi)^{101}\delta(\varphi'' R_\tau^{-1}+\varphi'_\tau)}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})^2}.
\end{aligned}
$$

That is:

$$I = \frac{\delta}{(1+\varphi'^2(s))^{1/2}}[-\varphi'\varphi_\tau - (1+\varphi'^2)R_\tau^{-1}] + \frac{100(\delta\xi)^{100}(\varphi'^2 R_\tau^{-1} + \varphi'\varphi_\tau)\delta}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})}$$
$$+\frac{(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta^2\xi}{(1+\varphi'(s)^2)(1+100(\delta\xi)^{100})} + \frac{\varphi' 10^4(\delta\xi)^{100}\delta(\varphi' R_\tau^{-1} + \varphi_\tau)}{(1+\varphi'^2(s))^{1/2}(1+100(\delta\xi)^{100})^2}$$
$$-\frac{\varphi'^2 10^4(\delta\xi)^{101}\delta(\varphi'' R_\tau^{-1} + \varphi'_\tau)}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})^2}, \tag{C.10}$$

$$II = \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial \tau}\frac{\partial x}{\partial s}$$
$$= \left(R_\tau^{-1} + \left(\frac{-\varphi'' R_\tau^{-1} - \varphi'_\tau}{(1+\varphi'^2(s))^{3/2}}\right)\frac{\delta\xi}{1+100(\delta\xi)^{100}}\right)\left(\varphi' R_s^{-1} - \frac{\varphi'\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\delta\xi\right)$$
$$-\left(\varphi' R_\tau^{-1} + \varphi_\tau - \frac{\varphi'(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^{3/2}}\right)$$
$$\times\left(R_s^{-1} + \left(\frac{-\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{1/2}} + \frac{\varphi'^2\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\right)\frac{\delta\xi}{1+100(\delta\xi)^{100}}\right)$$
$$= R_\tau^{-1}\varphi' R_s^{-1} - \frac{\varphi'\varphi'' R_s^{-1}R_\tau^{-1}\delta\xi}{(1+\varphi'^2(s))^{3/2}} - \frac{\varphi' R_s^{-1}(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})}$$
$$+\frac{(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta\xi\varphi'\varphi'' R_s^{-1}\delta\xi}{(1+\varphi'^2(s))^3(1+100(\delta\xi)^{100})} - (\varphi' R_\tau^{-1} + \varphi_\tau)R_s^{-1} + \frac{R_s^{-1}\varphi'(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta\xi}{(1+\varphi'^2(s))^{3/2}}$$
$$+\frac{(\varphi' R_\tau^{-1} + \varphi_\tau)\varphi'' R_s^{-1}\delta\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})} - \frac{\varphi'(\varphi'' R_\tau^{-1} + \varphi'_\tau)\delta\xi\varphi'' R_s^{-1}\delta\xi}{(1+\varphi'(s)^2)^3(1+100(\delta\xi)^{100})}$$
$$= -\varphi_\tau R_s^{-1} + \frac{\varphi' R_s^{-1}\varphi'_\tau\delta\xi}{(1+\varphi'^2(s))^{3/2}} + \frac{\delta\xi R_s^{-1}}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})}[\varphi''\varphi_\tau - \varphi'\varphi'_\tau]$$
$$= -\varphi_\tau\frac{L}{(1+\varphi'^2(s))^{1/2}} + \frac{\delta\xi\varphi'\varphi'_\tau L}{(1+\varphi'^2(s))^2} + \frac{\delta\xi L}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\varphi''\varphi_\tau$$
$$-\frac{\delta\xi L}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\varphi'\varphi'_\tau$$
$$= -\varphi_\tau\frac{L}{(1+\varphi'^2(s))^{1/2}} + \frac{\delta\xi L\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}$$
$$+\frac{\delta\xi\varphi'\varphi'_\tau L}{(1+\varphi'^2(s))^2}\left(1 - \frac{1}{1+100(\delta\xi)^{100}}\right),$$

and so:

$$II = -\varphi_\tau\frac{L}{(1+\varphi'^2(s))^{1/2}} + \frac{\delta\xi L\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}$$

$$+\frac{100(\delta\xi)^{101}L\varphi'\varphi'_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}. \tag{C.11}$$

In order to rewrite the α-equation in the appropriate form we need to calculate the time and spatial derivatives in the new variables. We first consider $\partial_t q$, which requires new estimates on the terms $\frac{I}{Det(s)}$ and $\frac{II}{Det(s)}$. When considering the limit equations the first of these terms, that is the coefficient of Ω_s, behaves well in the limit, and it can be written as an $O(1)$ term plus some error that vanishes in the limit as $\delta \to 0$. That is, for fixed $\xi \in \mathbb{R}$, using a geometric series as in the smooth case we note that:

$$\frac{1}{Det(s)} = \frac{1}{L\delta} + \frac{1}{L}\frac{\varphi''\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})} + o(1) \tag{C.12}$$

which when multiplied by I gives the following:

$$\frac{I}{Det(s)} = \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}[-\varphi'\varphi_\tau - (1+\varphi'^2)R_\tau^{-1}] + o(1).$$

As with the smooth case, the coefficient of Ω_ξ will contain a part of $O(\frac{1}{\delta})$ which, on analysing the terms that arise from $u \cdot \nabla q$, does not appear in the limit equations due to the sharp front equation (2.11). When considering the analytic case we consider $\xi \in \mathbb{R}$ and so terms of the form $\xi\Omega_\xi$ that appear, for example in (B.11), make further analysis of the equation more difficult. We show that no term of this form appears in the analytic case for $\alpha < 1$, for more details see Chapter 4. To show this, at present we do not consider estimates on $\frac{II}{Det(s)}$, and so the required form for the time derivative is:

$$\begin{aligned}
\partial_t q &= \frac{I}{Det(s)}\Omega_s + \frac{II}{Det(s)}\Omega_\xi + \Omega_\tau \\
&= \frac{1}{L}\frac{1}{(1+\varphi'^2(s))^{1/2}}[-\varphi'\varphi_\tau - (1+\varphi'^2)R_\tau^{-1}]\Omega_s \\
&\quad + \frac{1}{Det(s)}\Bigg(-\varphi_\tau\frac{L}{(1+\varphi'^2(s))^{1/2}} + \frac{\delta\xi L\varphi''\varphi_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})} \\
&\quad + \frac{100(\delta\xi)^{101}L\varphi'\varphi'_\tau}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\Bigg)\Omega_\xi \\
&\quad + \Omega_\tau + o(1).
\end{aligned} \tag{C.13}$$

The remainder of this chapter contains the calculations needed to obtain the appropriate form for $u \cdot \nabla q$. We first have that:

$$\partial_x q = \frac{1}{Det(s)}\frac{\partial y}{\partial \xi}\Omega_s - \frac{1}{Det(s)}\frac{\partial y}{\partial s}\Omega_\xi$$
$$= \frac{1}{Det(s)}\frac{\delta}{(1+\varphi'^2(s))^{1/2}}\Omega_s - \frac{1}{Det(s)}\left(\varphi' R_s^{-1} - \frac{\varphi'\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\delta\xi\right)\Omega_\xi$$
$$= \frac{1}{Det(s)}\frac{\delta}{(1+\varphi'^2(s))^{1/2}}\Omega_s$$
$$-\frac{1}{Det(s)}\left(\frac{\varphi' L}{(1+\varphi'^2(s))^{1/2}} - \frac{\varphi'\varphi'' L}{(1+\varphi'^2(s))^2)}\right)\Omega_\xi \tag{C.14}$$

and:

$$\partial_y q = \frac{1}{Det(s)}\frac{\partial x}{\partial s}\Omega_\xi - \frac{1}{Det(s)}\frac{\partial x}{\partial \xi}\Omega_s$$
$$= \frac{1}{Det(s)}\left(R_s^{-1} + \left(\frac{-\varphi'' R_s^{-1}}{(1+\varphi'^2(s))^{3/2}}\right)\frac{\delta\xi}{1+100(\delta\xi)^{100}}\right)\Omega_\xi$$
$$-\frac{1}{Det(s)}\left(-\frac{\varphi'}{(1+\varphi'^2(s))^{1/2}}\left(\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\right)\Omega_s$$
$$= \frac{1}{Det(s)}\left(\frac{L}{(1+\varphi'^2(s))^{1/2}} - \frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\right)\Omega_\xi$$
$$+\frac{1}{Det(s)}\left(\frac{\varphi'}{(1+\varphi'^2(s))^{1/2}}\left(\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\right)\Omega_s. \tag{C.15}$$

Recall that the unit normal and unit tangent vectors are given by:

$$\begin{cases} \mathbf{n}(s) = \left(\frac{-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}}, \frac{1}{(1+\varphi'^2(s))^{1/2}}\right) \\ \mathbf{t}(s) = \left(\frac{1}{(1+\varphi'^2(s))^{1/2}}, \frac{\varphi'(s)}{(1+\varphi'^2(s))^{1/2}}\right) \end{cases}$$

with the relations:

$$(1,0) = \frac{\mathbf{t} - \varphi'(s)\mathbf{n}}{(1+\varphi'^2(s))^{1/2}}, \tag{C.16}$$
$$(0,1) = \frac{\varphi'(s)\mathbf{t} + \mathbf{n}}{(1+\varphi'^2(s))^{1/2}}, \tag{C.17}$$

required for simplifying ∇q and $\nabla^\perp q$ that follow. Using the terms in (C.14) and (C.15) we have:

$$
\begin{aligned}
\nabla q &= (\partial_x q, \partial_y q) \\
&= \frac{\delta}{Det(s)}\mathbf{t}\Omega_s - (0,1)\frac{\delta}{Det(s)}\frac{\varphi'}{(1+\varphi'^2(s))^{1/2}}\Omega_s + \frac{L}{Det(s)}\mathbf{n}\Omega_\xi \\
&\quad -(0,1)\frac{L}{Det(s)}\frac{1}{(1+\varphi'^2(s))^{1/2}}\Omega_\xi - \frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^{3/2}}\mathbf{n}\Omega_\xi \\
&\quad +(0,1)\frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^2}\Omega_\xi \\
&\quad +\frac{1}{Det(s)}\left(\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{t}\Omega_s \\
&\quad -(1,0)\frac{1}{Det(s)}\frac{1}{(1+\varphi'^2(s))^{1/2}}\left(\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\Omega_s \\
&\quad +\frac{L}{Det(s)}\mathbf{n}\Omega_\xi + (1,0)\frac{L\varphi'}{Det(s)(1+\varphi'^2(s))^{1/2}}\Omega_\xi \\
&\quad -\frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})}\mathbf{n}\Omega_\xi \\
&\quad -(1,0)\frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100})}\Omega_\xi \\
&= \frac{\delta}{Det(s)}\mathbf{t}\Omega_s + \frac{L}{Det(s)}\mathbf{n}\Omega_\xi - \frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^{3/2}}\mathbf{n}\Omega_\xi \\
&\quad +\frac{1}{Det(s)}\left(\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{t}\Omega_s \\
&\quad -\frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^{3/2}(1+100(\delta\xi)^{100})}\mathbf{n}\Omega_\xi \\
&\quad -\frac{\varphi'\mathbf{t}}{(1+\varphi'^2(s))^{1/2}}\frac{\delta}{Det(s)}\frac{\varphi'}{(1+\varphi'^2(s))^{1/2}}\Omega_s - \frac{\mathbf{n}}{(1+\varphi'^2(s))^{1/2}}\frac{\delta}{Det(s)}\frac{\varphi'}{(1+\varphi'^2(s))^{1/2}}\Omega_s \\
&\quad +\frac{\varphi'\mathbf{t}}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^2}\Omega_\xi + \frac{\mathbf{n}}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^2}\Omega_\xi \\
&\quad -\frac{\mathbf{t}}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\frac{1}{(1+\varphi'^2(s))^{1/2}}\left(\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\Omega_s \\
&\quad +\frac{\varphi'\mathbf{n}}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\frac{1}{(1+\varphi'^2(s))^{1/2}}\left(\frac{\delta}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\Omega_s \\
&\quad -\frac{\mathbf{t}}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\frac{\varphi'\varphi'' L\delta\xi}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100}}\Omega_\xi \\
&\quad +\frac{\varphi'\mathbf{n}}{(1+\varphi'^2(s))^{1/2}}\frac{1}{Det(s)}\frac{\varphi'\varphi'' L\delta\xi}{(1+\varphi'^2(s))^2(1+100(\delta\xi)^{100}}\Omega_\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\delta}{Det(s)}\mathbf{t}\Omega_s + \frac{L}{Det(s)}\mathbf{n}\Omega_\xi - \frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^{3/2}}\mathbf{n}\Omega_\xi \\
&+\frac{\varphi'(s)^2}{(1+\varphi'^2(s))}\frac{1}{Det(s)}\left(\frac{-\delta 100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{t}\Omega_s \\
&+\frac{\varphi'(s)}{(1+\varphi'^2(s))}\frac{1}{Det(s)}\left(\frac{-\delta 100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}} - \frac{10^4(\delta\xi)^{100}\delta}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{n}\Omega_s \\
&+\frac{1}{Det(s)}\frac{\varphi'\varphi'' L\delta\xi}{(1+\varphi'^2(s))^{5/2}}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}}\right)\mathbf{t}\Omega_\xi \\
&+\frac{1}{Det(s)}\frac{\varphi'' L\delta\xi}{(1+\varphi'^2(s))^{5/2}}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}}\right)\mathbf{n}\Omega_\xi
\end{aligned}
$$

giving:

$$
\begin{aligned}
\nabla q &= \frac{\delta}{Det(s)}\mathbf{t}(s)\Omega_s + \frac{L}{Det(s)}\mathbf{n}(s)\Omega_\xi - \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}}\frac{\delta}{Det(s)}\mathbf{n}(s)\Omega_\xi \\
&-\frac{\varphi'}{(1+\varphi'^2(s))}\frac{\delta}{Det(s)}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}} + \frac{10^4(\delta\xi)^{100}}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{n}(s)\Omega_s \\
&-\frac{\varphi'^2}{(1+\varphi'^2(s))}\frac{\delta}{Det(s)}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}} + \frac{10^4(\delta\xi)^{100}}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{t}(s)\Omega_s \\
&+\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta}{Det(s)}\frac{\varphi'' L}{(1+\varphi'^2(s))^{5/2}}\mathbf{n}(s)\Omega_\xi \\
&+\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta}{Det(s)}\frac{\varphi'\varphi'' L}{(1+\varphi'^2(s))^{5/2}}\mathbf{t}(s)\Omega_\xi. \qquad \text{(C.18)}
\end{aligned}
$$

Using the relations $\mathbf{n}^\perp = -\mathbf{t}$, $\mathbf{t}^\perp = \mathbf{n}$ we obtain the corresponding term:

$$
\begin{aligned}
\nabla^\perp q &= \frac{\delta}{Det(s)}\mathbf{n}(s)\Omega_s - \frac{L}{Det(s)}\mathbf{t}(s)\Omega_\xi + \frac{\varphi'' L\xi}{(1+\varphi'^2(s))^{3/2}}\frac{\delta}{Det(s)}\mathbf{t}(s)\Omega_\xi \\
&+\frac{\varphi'}{(1+\varphi'^2(s))}\frac{\delta}{Det(s)}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}} + \frac{10^4(\delta\xi)^{100}}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{t}(s)\Omega_s \\
&-\frac{\varphi'^2}{(1+\varphi'^2(s))}\frac{\delta}{Det(s)}\left(\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}} + \frac{10^4(\delta\xi)^{100}}{(1+100(\delta\xi)^{100})^2}\right)\mathbf{n}(s)\Omega_s \\
&-\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta}{Det(s)}\frac{\varphi'' L}{(1+\varphi'^2(s))^{5/2}}\mathbf{t}(s)\Omega_\xi \\
&+\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta}{Det(s)}\frac{\varphi'\varphi'' L}{(1+\varphi'^2(s))^{5/2}}\mathbf{n}(s)\Omega_\xi. \qquad \text{(C.19)}
\end{aligned}
$$

To study the term $u \cdot \nabla q$, we utilise these results and recall that:

$$u(x,y,t) = \iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \frac{1}{(\cosh(y-\bar{y}) - \cos(x-\bar{x})^{\alpha/2}} \nabla^{\perp}_{\bar{x},\bar{y}} q(\bar{x},\bar{y}) \mathrm{d}\bar{x}\mathrm{d}\bar{y} \tag{C.20}$$

which under the change of coordinates as defined in (C.2) is given by:

$$u(s,\xi,\tau)\cdot\nabla q = \iint\limits_{\mathbb{R}/\mathbb{Z}\times\mathbb{R}} \tilde{K}^{\delta}_{\alpha}(s,\bar{s},\xi,\bar{\xi}) \nabla^{\perp}_{\bar{s},\bar{\xi}}\Omega(\bar{s},\bar{\xi}) \cdot \nabla_{s,\xi}\Omega(s,\xi) Det(\bar{s}) \mathrm{d}\bar{s}\mathrm{d}\bar{\xi}, \tag{C.21}$$

where $\tilde{K}^{\delta}_{\alpha}(s,\bar{s},\xi,\bar{\xi})$ is as previously defined in Chapter 4. We also require the following relations that remain unchanged from Appendix B:

$$\begin{aligned}
\mathbf{t}(s)\cdot\mathbf{t}(\bar{s}) = \frac{(1,\varphi'(s))}{(1+\varphi'^2(s))^{1/2}} \cdot \frac{(1,\varphi'(\bar{s}))}{(1+\varphi'^2(\bar{s}))^{1/2}} &= \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}, \\
\mathbf{n}(s)\cdot\mathbf{n}(\bar{s}) = \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} &= \mathbf{t}(s)\cdot\mathbf{t}(\bar{s}), \\
\mathbf{t}(s)\cdot\mathbf{n}(\bar{s}) &= \frac{\varphi'(s)-\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}}, \\
\mathbf{n}(s)\cdot\mathbf{t}(\bar{s}) = \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s}))^{1/2}} &= -\mathbf{t}(s)\cdot\mathbf{n}(\bar{s}).
\end{aligned}$$

The terms that arise from taking the product $\nabla^{\perp}_{\bar{s},\bar{\xi}}\Omega(\bar{s},\bar{\xi}) \cdot \nabla_{s,\xi}\Omega(s,\xi)Det(\bar{s})$ are lengthy, yet contain common terms (a characterisation of the terms that occur is given in Chapter 5). We introduce some notation here in order to simplify the terms that follow, that is define:

$$A(\xi) = \frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}} + \frac{10^4(\delta\xi)^{100}}{(1+100(\delta\xi)^{100})^2}, \tag{C.22}$$

$$J_{-}(\bar{s}) = \frac{\varphi'(\bar{s})-\varphi'(s)}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s})^{1/2}}, \tag{C.23}$$

$$J_{+}(\bar{s}) = \frac{1+\varphi'(s)\varphi'(\bar{s})}{(1+\varphi'^2(s))^{1/2}(1+\varphi'^2(\bar{s})^{1/2}}, \tag{C.24}$$

then, using the previous results, we have precisely:

$$
\begin{aligned}
&\nabla^{\perp}_{\bar{s},\bar{\xi}}\Omega(\bar{s},\bar{\xi})\cdot\nabla_{s,\xi}\Omega(s,\xi)Det(\bar{s})\\
&= -\frac{\delta^2}{Det(s)}J_-(\bar{s})\Omega_{\bar{s}}\Omega_s + \frac{\delta L}{Det(s)}J_+(\bar{s})\Omega_{\bar{s}}\Omega_\xi - \frac{\varphi''(s)L\xi\delta^2}{(1+\varphi'^2(s))^{3/2}Det(s)}J_+(\bar{s})\Omega_{\bar{s}}\Omega_\xi\\
&\quad -\frac{\varphi'(s)\delta^2}{(1+\varphi'^2(s))Det(s)}A(\xi)J_+(\bar{s})\Omega_{\bar{s}}\Omega_s - \frac{\varphi'(s)^2\delta^2}{(1+\varphi'^2(s))Det(s)}A(\xi)J_-(\bar{s})\Omega_{\bar{s}}\Omega_s\\
&\quad +\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi''(s)L}{(1+\varphi'^2(s))^{5/2}}J_+(\bar{s})\Omega_{\bar{s}}\Omega_\xi\\
&\quad -\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'(s)\varphi''(s)L}{(1+\varphi'^2(s))^{5/2}}J_-(\bar{s})\Omega_{\bar{s}}\Omega_\xi\\
&\quad -\frac{L^2}{Det(s)}J_-(\bar{s})\Omega_\xi\Omega_{\bar{\xi}} - \frac{L\delta}{Det(s)}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_s + \frac{\varphi''(s)L^2\xi\delta}{(1+\varphi'^2(s))^{3/2}Det(s)}J_-(\bar{s})\Omega_\xi\Omega_{\bar{\xi}}\\
&\quad +\frac{\varphi'(s)\delta L}{(1+\varphi'^2(s))Det(s)}A(\xi)J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_s + \frac{\varphi'(s)^2\delta L}{(1+\varphi'^2(s))Det(s)}A(\xi)J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_s\\
&\quad -\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}}\frac{\delta}{Det(s)}\frac{\varphi''(s)L^2}{(1+\varphi'^2(s))^{5/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\\
&\quad -\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}}\frac{\delta}{Det(s)}\frac{\varphi'(s)\varphi''(s)L^2}{(1+\varphi'^2(s))^{5/2}}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\\
&\quad +\frac{\varphi''(\bar{s})L\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\frac{\delta^2}{Det(s)}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_s + \frac{\varphi''(\bar{s})L^2\bar{\xi}\delta}{(1+\varphi'^2(\bar{s}))^{3/2}Det(s)}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\\
&\quad -\frac{\varphi''(\bar{s})\varphi''(s)L^2\delta^2\bar{\xi}\xi}{(1+\varphi'^2(s))^{3/2}(1+\varphi'^2(\bar{s}))^{3/2}Det(s)}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\\
&\quad -\frac{\varphi''(\bar{s})L\bar{\xi}\varphi'(s)\delta^2}{(1+\varphi'^2(s))(1+\varphi'^2(\bar{s}))^{3/2}Det(s)}A(\xi)J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_s\\
&\quad -\frac{\varphi''(\bar{s})L\bar{\xi}\varphi'^2(s)\delta^2}{(1+\varphi'^2(s))(1+\varphi'^2(\bar{s}))^{3/2}Det(s)}A(\xi)J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_s\\
&\quad +\frac{100(\delta\xi)^{100}\xi\bar{\xi}}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi''(\bar{s})\varphi''(s)L^2}{(1+\varphi'^2(\bar{s}))^{3/2}(1+\varphi'^2(s))^{5/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\\
&\quad +\frac{100(\delta\xi)^{100}\xi\bar{\xi}}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi''(\bar{s})\varphi'(s)\varphi''(s)L^2}{(1+\varphi'^2(\bar{s}))^{3/2}(1+\varphi'^2(s))^{5/2}}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\\
&\quad +\frac{\varphi'(\bar{s})\delta^2}{(1+\varphi'^2(\bar{s}))Det(s)}A(\bar{\xi})J_+(\bar{s})\Omega_{\bar{s}}\Omega_s + \frac{\varphi'(\bar{s})\delta L}{(1+\varphi'^2(\bar{s}))Det(s)}A(\bar{\xi})J_-(\bar{s})\Omega_{\bar{s}}\Omega_\xi\\
&\quad -\frac{\varphi'(\bar{s})\varphi''(s)L\xi\delta^2}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))^{3/2}Det(s)}A(\bar{\xi})J_-(\bar{s})\Omega_{\bar{s}}\Omega_\xi\\
&\quad -\frac{\varphi'(\bar{s})\varphi'(s)\delta^2}{(1+\varphi'(\bar{s})^2)(1+\varphi'(s)^2)Det(s)}A(\bar{\xi})A(\xi)J_-(\bar{s})\Omega_{\bar{s}}\Omega_s\\
&\quad -\frac{\varphi'(\bar{s})\varphi'(s)^2\delta^2}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))Det(s)}A(\bar{\xi})A(\xi)J_+(\bar{s})\Omega_{\bar{s}}\Omega_s
\end{aligned}
$$

$$+\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'(\bar{s})}{(1+\varphi'^2(\bar{s}))}\frac{\varphi''(s)L}{(1+\varphi'^2(s))^{5/2}}A(\bar{\xi})J_-(\bar{s})\Omega_{\bar{s}}\Omega_\xi$$

$$+\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'(\bar{s})}{(1+\varphi'^2(\bar{s}))}\frac{\varphi'(s)\varphi''(s)L}{(1+\varphi'^2(s))^{5/2}}A(\bar{\xi})J_+(\bar{s})\Omega_{\bar{s}}\Omega_\xi$$

$$+\frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))}\frac{\delta^2}{Det(s)}A(\bar{\xi})J_-(\bar{s})\Omega_{\bar{s}}\Omega_s-\frac{\varphi'^2(\bar{s})\delta L}{(1+\varphi'^2(\bar{s}))Det(s)}A(\bar{\xi})J_+(\bar{s})\Omega_{\bar{s}}\Omega_\xi$$

$$+\frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))}\frac{\delta^2}{Det(s)}\frac{\varphi''(s)L\xi}{(1+\varphi'^2(s))^{3/2}}A(\bar{\xi})J_+(\bar{s})\Omega_{\bar{s}}\Omega_\xi$$

$$+\frac{\varphi'^2(\bar{s})\varphi'(s)\delta^2}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))Det(s)}A(\bar{\xi})A(\xi)J_+(\bar{s})\Omega_{\bar{s}}\Omega_s$$

$$-\frac{\varphi'^2(\bar{s})\varphi'(s)^2\delta^2}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))Det(s)}A(\bar{\xi})A(\xi)J_-(\bar{s})\Omega_{\bar{s}}\Omega_s$$

$$-\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))}A(\bar{\xi})\frac{\varphi''(s)L}{(1+\varphi'^2(s))^{5/2}}J_+(\bar{s})\Omega_{\bar{s}}\Omega_\xi$$

$$+\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))}A(\bar{\xi})\frac{\varphi'(s)\varphi''(s)L}{(1+\varphi'^2(s))^{5/2}}J_-(\bar{s})\Omega_{\bar{s}}\Omega_\xi$$

$$-\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi''(\bar{s})L}{(1+\varphi'^2(\bar{s}))^{5/2}}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_s$$

$$-\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\delta}{Det(s)}\frac{\varphi''(\bar{s})L^2}{(1+\varphi'^2(\bar{s}))^{5/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi$$

$$+\frac{100(\delta\bar{\xi})^{100}\bar{\xi}\xi}{1+100(\delta\bar{\xi})^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi''(\bar{s})\varphi''(s)L^2}{(1+\varphi'^2(\bar{s}))^{5/2}(1+\varphi'^2(s))^{3/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi$$

$$+\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi''(\bar{s})\varphi'(s)L}{(1+\varphi'^2(\bar{s}))^{5/2}(1+\varphi'^2(s))}A(\xi)J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_s$$

$$+\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi''(\bar{s})\varphi'^2(s)L}{(1+\varphi'^2(\bar{s}))^{5/2}(1+\varphi'^2(s))}A(\xi)J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_s$$

$$-\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi''(s)\varphi''(\bar{s})L^2}{(1+\varphi'^2(s))^{5/2}(1+\varphi'^2(\bar{s}))^{5/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi$$

$$-\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'(s)\varphi''(s)\varphi''(\bar{s})L^2}{(1+\varphi'^2(s))^{5/2}(1+\varphi'^2(\bar{s}))^{5/2}}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi$$

$$-\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\varphi'(\bar{s})\varphi''(\bar{s})L}{(1+\varphi'^2(\bar{s}))^{5/2}}\frac{\delta^2}{Det(s)}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_s$$

$$+\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\delta}{Det(s)}\frac{\varphi'(\bar{s})\varphi''(\bar{s})L^2}{(1+\varphi'^2(\bar{s}))^{5/2}}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi$$

$$-\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'(\bar{s})\varphi''(\bar{s})L^2\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}(1+\varphi'^2(\bar{s}))^{5/2}}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi$$

$$-\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'(s)}{(1+\varphi'^2(s))}\frac{\varphi'(\bar{s})\varphi''(\bar{s})L}{(1+\varphi'^2(\bar{s}))^{5/2}}A(\xi)J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_s$$

$$+\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'^2(s)}{(1+\varphi'^2(s))}\frac{\varphi'(\bar{s})\varphi''(\bar{s})L}{(1+\varphi'^2(\bar{s}))^{5/2}}A(\xi)J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_s$$

$$+\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'(\bar{s})\varphi''(\bar{s})\varphi''(s)L^2}{(1+\varphi'^2(s))^{5/2}(1+\varphi'^2(\bar{s}))^{5/2}}J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi$$
$$-\frac{100(\delta\bar{\xi})^{100}\bar{\xi}}{1+100(\delta\bar{\xi})^{100}}\frac{100(\delta\xi)^{100}\xi}{1+100(\delta\xi)^{100}}\frac{\delta^2}{Det(s)}\frac{\varphi'(\bar{s})\varphi''(\bar{s})\varphi'(s)\varphi''(s)L^2}{(1+\varphi'^2(s))^{5/2}(1+\varphi'^2(\bar{s}))^{5/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi.$$

Introducing the term:

$$B(\xi)=\frac{100(\delta\xi)^{100}}{1+100(\delta\xi)^{100}} \tag{C.25}$$

to further simplify the equations, we obtain the final form for the integral terms, where the double integrals are taken over the domain $(\bar{s},\bar{\xi})\in\mathbb{R}/\mathbb{Z}\times\mathbb{R}$:

$$u\cdot\nabla q \tag{C.26}$$
$$=-\frac{\delta^2}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\Omega_{\bar{s}}\Omega_s\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.27}$$
$$+\frac{\delta L}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})(\Omega_{\bar{s}}\Omega_\xi-\Omega_{\bar{\xi}}\Omega_s)\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.28}$$
$$+\frac{\delta^2 L}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\left[\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\Omega_{\bar{\xi}}\Omega_s-\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}\Omega_{\bar{s}}\Omega_\xi\right]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.29}$$
$$+\frac{\delta L^2}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\left[\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{3/2}}+\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{3/2}}\right]\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.30}$$
$$-\frac{\delta^2}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})\frac{\varphi''(\bar{s})\varphi''(s)\bar{\xi}\xi}{(1+\varphi'^2(s))^{3/2}(1+\varphi'^2(\bar{s}))^{3/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.31}$$
$$-\frac{L^2}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\Omega_\xi\Omega_{\bar{\xi}}\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.32}$$
$$-\frac{\delta^2}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\left[\frac{\varphi'(s)}{(1+\varphi'^2(s))}A(\xi)-\frac{\varphi'(\bar{s})}{(1+\varphi'^2(\bar{s}))}A(\bar{\xi})\right]\Omega_{\bar{s}}\Omega_s\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.33}$$
$$-\frac{\delta^2}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\left[\frac{\varphi'^2(s)}{(1+\varphi'^2(s))}A(\xi)-\frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))}A(\bar{\xi})\right]\Omega_{\bar{s}}\Omega_s\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.34}$$
$$+\frac{\delta^2 L}{Det(s)}\iint\tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\left[\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{5/2}}B(\xi)\Omega_{\bar{s}}\Omega_\xi-\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{5/2}}B(\bar{\xi})\Omega_{\bar{\xi}}\Omega_s\right]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.35}$$

$$-\frac{\delta^2 L}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\Bigg[\frac{\varphi'(s)\varphi''(s)\xi}{(1+\varphi'^2(s))^{5/2}}B(\xi)\Omega_{\bar{s}}\Omega_\xi$$

$$+\frac{\varphi'(\bar{s})\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{5/2}}B(\bar{\xi})\Omega_{\bar{\xi}}\Omega_s\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.36}$$

$$+\frac{\delta L}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\Bigg[\frac{\varphi'(s)}{(1+\varphi'^2(s))}A(\xi)\Omega_{\bar{\xi}}\Omega_s$$

$$+\frac{\varphi'(\bar{s})}{(1+\varphi'^2(\bar{s}))}A(\bar{\xi})\Omega_{\bar{s}}\Omega_\xi\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.37}$$

$$+\frac{\delta L}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\Bigg[\frac{\varphi'^2(s)}{(1+\varphi'^2(s))}A(\xi)\Omega_{\bar{\xi}}\Omega_s$$

$$-\frac{\varphi'^2(\bar{s})}{(1+\varphi'^2(\bar{s}))}A(\bar{\xi})\Omega_{\bar{s}}\Omega_\xi\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.38}$$

$$-\frac{\delta L^2}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\Bigg[\frac{\varphi''(s)\xi}{(1+\varphi'^2(s))^{5/2}}B(\xi)$$

$$+\frac{\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{5/2}}B(\bar{\xi})\Bigg]\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.39}$$

$$-\frac{\delta L^2}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\Bigg[\frac{\varphi'(s)\varphi''(s)\xi}{(1+\varphi'^2(s))^{5/2}}B(\xi)$$

$$+\frac{\varphi'(\bar{s})\varphi''(\bar{s})\bar{\xi}}{(1+\varphi'^2(\bar{s}))^{5/2}}B(\bar{\xi})\Bigg]\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.40}$$

$$-\frac{\delta^2 L}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\Bigg[\frac{\varphi''(\bar{s})\varphi'(s)\bar{\xi}}{(1+\varphi'^2(s))(1+\varphi'^2(\bar{s}))^{3/2}}A(\xi)\Omega_{\bar{\xi}}\Omega_s$$

$$+\frac{\varphi''(s)\varphi'(\bar{s})\xi}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))^{3/2}}A(\bar{\xi})\Omega_{\bar{s}}\Omega_\xi\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.41}$$

$$-\frac{\delta^2 L}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\Bigg[\frac{\varphi''(\bar{s})\varphi'^2(s)\bar{\xi}}{(1+\varphi'^2(s))(1+\varphi'^2(\bar{s}))^{3/2}}A(\xi)\Omega_{\bar{\xi}}\Omega_s$$

$$-\frac{\varphi''(s)\varphi'^2(\bar{s})\xi}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))^{3/2}}A(\bar{\xi})\Omega_{\bar{s}}\Omega_\xi\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.42}$$

$$+\frac{\delta^2 L^2}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\varphi''(\bar{s})\varphi''(s)\bar{\xi}\xi\Bigg[\frac{1}{(1+\varphi'^2(\bar{s}))^{3/2}(1+\varphi'^2(s))^{5/2}}B(\xi)$$

$$+\frac{1}{(1+\varphi'^2(s))^{3/2}(1+\varphi'^2(\bar{s}))^{5/2}}B(\bar{\xi})\Bigg]\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.43}$$

$$+\frac{\delta^2 L^2}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\varphi''(\bar{s})\varphi''(s)\bar{\xi}\xi\Bigg[\frac{\varphi'(s)}{(1+\varphi'^2(\bar{s}))^{3/2}(1+\varphi'^2(s))^{5/2}}B(\xi)$$

$$-\frac{\varphi'(\bar{s})}{(1+\varphi'^2(s))^{3/2}(1+\varphi'^2(\bar{s}))^{5/2}}B(\bar{\xi})\Bigg]\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.44}$$

$$-\frac{\delta^2}{Det(s)}\iint \tilde{K}_\alpha^\delta(s,\bar{s},\xi,\bar{\xi})A(\xi)A(\bar{\xi})J_-(\bar{s})\frac{\varphi'(\bar{s})\varphi'(s)}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))}\Omega_{\bar{s}}\Omega_s\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.45}$$

$$+\frac{\delta^2}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})A(\bar{\xi})A(\xi)$$
$$\times\frac{\varphi'(\bar{s})\varphi'(s)}{(1+\varphi'^2(\bar{s}))^{1/2}(1+\varphi'^2(s))^{1/2}}J_-(\bar{s})J_+(\bar{s})\Omega_{\bar{s}}\Omega_s\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.46}$$
$$+\frac{\delta^2 L}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\Bigg[\frac{\xi\varphi'(\bar{s})\varphi''(s)}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))^{5/2}}A(\bar{\xi})B(\xi)\Omega_{\bar{s}}\Omega_\xi$$
$$+\frac{\bar{\xi}\varphi'(s)\varphi''(\bar{s})}{(1+\varphi'^2(s))(1+\varphi'^2(\bar{s}))^{5/2}}A(\xi)B(\bar{\xi})\Omega_{\bar{\xi}}\Omega_s\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.47}$$
$$+\frac{\delta^2 L}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\Bigg[\frac{\xi\varphi'(\bar{s})\varphi'(s)\varphi''(s)}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))^{5/2}}A(\bar{\xi})B(\xi)\Omega_{\bar{s}}\Omega_\xi$$
$$-\frac{\bar{\xi}\varphi'(s)\varphi'(\bar{s})\varphi''(\bar{s})}{(1+\varphi'^2(s))(1+\varphi'^2(\bar{s}))^{5/2}}A(\xi)B(\bar{\xi})\Omega_{\bar{\xi}}\Omega_s\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.48}$$
$$-\frac{\delta^2}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})\frac{\varphi'^2(\bar{s})\varphi'^2(s)}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))}A(\bar{\xi})A(\xi)J_-(\bar{s})\Omega_{\bar{s}}\Omega_s\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.49}$$
$$-\frac{\delta^2 L}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_+(\bar{s})\Bigg[\frac{\xi\varphi'^2(\bar{s})\varphi''(s)}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))^{5/2}}A(\bar{\xi})B(\xi)\Omega_{\bar{s}}\Omega_\xi$$
$$-\frac{\bar{\xi}\varphi'^2(s)\varphi''(\bar{s})}{(1+\varphi'^2(s))(1+\varphi'^2(\bar{s}))^{5/2}}A(\xi)B(\bar{\xi})\Omega_{\bar{\xi}}\Omega_s\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.50}$$
$$+\frac{\delta^2 L}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})J_-(\bar{s})\Bigg[\frac{\xi\varphi'^2(\bar{s})\varphi'(s)\varphi''(s)}{(1+\varphi'^2(\bar{s}))(1+\varphi'^2(s))^{5/2}}A(\bar{\xi})B(\xi)\Omega_{\bar{s}}\Omega_\xi$$
$$+\frac{\bar{\xi}\varphi'^2(s)\varphi'(\bar{s})\varphi''(\bar{s})}{(1+\varphi'^2(s))(1+\varphi'^2(\bar{s}))^{5/2}}A(\xi)B(\bar{\xi})\Omega_{\bar{\xi}}\Omega_s\Bigg]\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.51}$$
$$-\frac{\delta^2 L^2}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})B(\xi)B(\bar{\xi})\frac{\xi\bar{\xi}\varphi''(s)\varphi''(\bar{s})}{(1+\varphi'^2(s))^{5/2}(1+\varphi'^2(\bar{s}))^{5/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.52}$$
$$+\frac{\delta^2 L^2}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})B(\xi)B(\bar{\xi})\frac{\xi\bar{\xi}\varphi''(s)\varphi''(\bar{s})}{(1+\varphi'^2(s))^2(1+\varphi'^2(\bar{s}))^2}J_-(\bar{s})J_+(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi} \tag{C.53}$$
$$-\frac{\delta^2 L^2}{Det(s)}\iint \tilde{K}^\delta_\alpha(s,\bar{s},\xi,\bar{\xi})B(\xi)B(\bar{\xi})\frac{\xi\bar{\xi}\varphi'(s)\varphi''(s)\varphi'(\bar{s})\varphi''(\bar{s})}{(1+\varphi'^2(s))^{5/2}(1+\varphi'^2(\bar{s}))^{5/2}}J_-(\bar{s})\Omega_{\bar{\xi}}\Omega_\xi\mathrm{d}\bar{s}\mathrm{d}\bar{\xi}. \tag{C.54}$$

In §5.1 the terms presented in (C.13) and (C.27) - (C.54) are classified into like terms in order to rewrite the α-equation in a more general form in the new coordinates; this is given in (5.25) and (5.27). In deriving this equation, in chapter 5, we study the product $G \cdot U$ (see pages 80/81). In order to classify the terms and to see which need further analysis it is necessary to count the powers of δ and ξ that appear. The combinations that appear are shown in the following table for quick reference.

Table C.1: $U \cdot G$

	U_1, U_4, U_5 $\delta\Omega_{\bar{s}}$	U_2 $\Omega_{\bar{\xi}}$	U_3, U_6, U_7 $\delta\bar{\xi}\Omega_{\bar{\xi}}$
G_1, G_4, G_5 Ω_s	$\delta\Omega_{\bar{s}}\Omega_s$	$\Omega_{\bar{\xi}}\Omega_s$	$\delta\bar{\xi}\Omega_{\bar{\xi}}\Omega_s$
	C.27, C.33, C.34, C.45, C.46, C.49	C.28, C.37, C.38	C.29, C.35, C.36, C.41, C.42, C.47, C.48, C.50, C.51
G_2 $\frac{1}{Det(s)}\Omega_\xi$	$\Omega_{\bar{s}}\Omega_\xi$	$\Omega_{\bar{\xi}}\xi\Omega_\xi$	$\bar{\xi}\Omega_{\bar{\xi}}\Omega_\xi$
	C.28, C.37, C.38	C.32	C.30, C.39, C.40
G_3, G_6, G_7 $\xi\Omega_\xi$	$\delta\Omega_{\bar{s}}\xi\Omega_\xi$	$\Omega_{\bar{\xi}}\xi\Omega_\xi$	$\delta\bar{\xi}\Omega_{\bar{\xi}}\xi\Omega_\xi$
	C.29, C.35, C.36, C.41, C.42, C.47, C.48, C.50, C.51	C.30, C.39, C.40	C.31, C.43, C.44, C.52, C.53, C.54

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